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Stochastic Analysis of Unsaturated Flow in Heterogeneous Soils
2. Statistically Anisotropic Media With Variable α

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Steady unsaturated flow in heterogeneous soil with an arbitrarily oriented mean hydraulic gradient is analyzed using spectral solutions of the stochastic perturbation equation which describes the capillary pressure head ψ. The unsaturated hydraulic conductivity is related to ψ by $K = K_s \exp(-\alpha\psi)$, where K_s is the saturated hydraulic conductivity and α is a soil parameter, and both K_s and α are treated as three-dimensional statistically homogeneous, anisotropic random fields. Analytical results are obtained for the capillary pressure head variance and the effective (mean) unsaturated hydraulic conductivity. The head variance depends upon the degree of anisotropy of the ln K_s covariance; when α is random, the head variance increases with mean capillary pressure head. The effective hydraulic conductivity for arbitrary orientation of the mean hydraulic gradient $\mathbf{J}$ is shown to have tensorial properties, but its components depend on the magnitude and direction of $\mathbf{J}$ and the orientation of the stratification in the soil. When α is random, the degree of anisotropy of the effective conductivity depends strongly on mean capillary pressure.

Introduction

In part 1 of this series of papers, we demonstrated the importance of dimensionality in the analysis of flow in statistically isotropic soil. However, natural soils or geological formations, in general, exhibit bedding or stratification. Because of this geological structure, hydrologic properties of the porous media should be considered to be statistically anisotropic. Statistical anisotropy implies that the correlation scale of a stochastic process depends on the direction. Hydraulic conductivity, for example, usually correlates at longer distance in the direction parallel to bedding than in the direction perpendicular to bedding. Saturated hydraulic conductivity data presented by *Smith* [1978] from both horizontal and vertical profiles of the Quadra sand outcrop in Vancouver, British Columbia indicate that the correlation scale of the conductivity in the horizontal profile is an order of magnitude larger than that of the vertical. *Byer and Stephens* [1983] also observed statistical anisotropy in hydraulic conductivity data from alluvial sands. Thus it is necessary to extend the analysis to statistically anisotropic media in order to more realistically describe the effects of field heterogeneity.

In the assumed exponential $K - \psi$ relationship (see equation (1) *Yeh et al.* [this issue] referred to as part 1 in the following) the saturated hydraulic conductivity K_s and the parameter α are the two parameters characterizing the unsaturated hydraulic conductivity behavior as a function of capillary pressure. The effects of spatially variable saturated hydraulic conductivity on unsaturated flow have been analyzed in the part 1 of this paper. Although field variation of the parameter α has not been as fully investigated as the saturated hydraulic conductivity, variation of similar parameters which control the rate of reduction in unsaturated hydraulic conductivity have been reported by *Warrick et al.* [1977], *Russo and Bresler* [1981], and *Yeh* [1982]. To be more realistic, one should consider the effect of the randomness of α in the analysis of unsaturated flow in heterogeneous soils.

In the first section of this paper the effects of statistical anisotropy or stratification on head variance and effective hydraulic conductivity will be investigated for the media with spatially variable saturated hydraulic conductivity and a constant α parameter. Then the full tensorial property of effective hydraulic conductivity will be derived. Finally, the effective hydraulic conductivities of media in which both K_s and α are regarded as stochastic processes will be developed.

Effects of Statistical Anisotropy

In order to focus on the effects of statistical anisotropy on unsaturated flow, we first consider the case where α is a deterministic constant. The mean hydraulic gradient is assumed to exist in the vertical direction x_1 only ($J_2 = J_3 = 0$). Thus for a three-dimensional steady state flow the spectral solution to the general stochastic partial differential equation (part 1, equation (10)) reduces to

$$S_{hh} = \frac{J_1^2 k_1^2 S_{ff}}{(k^4 + \beta^2 k_1^2)} \qquad (1)$$

where $\beta = \alpha(2J_1 - 1)$.

To evaluate (1) a three-dimensional anisotropic covariance function of the saturated hydraulic conductivity random field is adopted, following *Gelhar and Axness* [1983], in the form

$$R_{ff}(\xi) = \sigma_f^2 \exp\left[-\left(\frac{\xi_1^2}{\lambda_1^2} + \frac{\xi_2^2}{\lambda_2^2} + \frac{\xi_3^2}{\lambda_3^2}\right)^{1/2}\right] \qquad (2)$$

where λ_1, λ_2, and λ_3 are the correlation scales in the x_1, x_2, and x_3 directions. The spectrum S_{ff} corresponding to this covariance function is

$$S_{ff} = \frac{\sigma_f^2 \lambda_1 \lambda_2 \lambda_3}{\pi^2(1 + \lambda_1^2 k_1^2 + \lambda_2^2 k_2^2 + \lambda_3^2 k_3^2)^2} \qquad (3)$$

Substitution of this spectrum in (1) results in the spectrum S_{hh} of capillary pressure head

$$S_{hh} = J_1^2 \sigma_f^2 \lambda_1 \lambda_2 \lambda_3 k_1^2 \cdot (\pi(k^4 + \beta^2 k_1^2)(1 + \lambda_1^2 k_1^2 + \lambda_2^2 k_2^2 + \lambda_3^2 k_3^2)^2)^{-1} \qquad (4)$$

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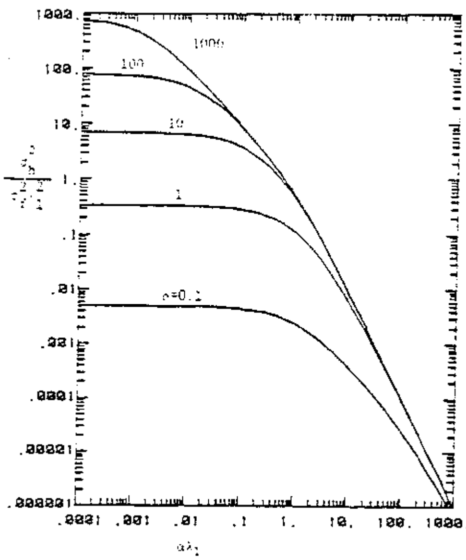

Fig. 1. Variances of capillary pressure resulting from three-dimensional anisotropic ln K_s field with different aspect ratios; $\rho = \lambda/\lambda_1$ and $J = 1$.

Further, assume that the hydraulic conductivity variation is statistically isotropic in the horizontal plane, i.e., $\lambda_2 = \lambda_3 = \lambda$. The capillary pressure head variance σ_h^2 is obtained by integrating the spectrum (4) over wave number space (see the appendix). The general form of the head variance is

$$\sigma_h^2 = J_1^2\sigma_f^2\lambda_1^2\rho^4 \int_0^1 \frac{t^2\,dt}{[(\rho^2-1)t^2+\rho^2 gt+1]^2} \qquad (5)$$

where $\rho = \lambda/\lambda_1$ is the aspect ratio, $g = \lambda_1\beta$ and $\beta = \alpha/(2J_1 - 1)$. After integrating (5), the capillary pressure head variance becomes

$$\sigma_h^2 = J_1^2\sigma_f^2\lambda_1^2\rho^4\left[\frac{(\rho^2 g+2)}{[\rho^4 g^2-4(\rho^2-1)](\rho^2+\rho^2 g)} + \frac{2}{[\rho^4 g^2-4(\rho^2-1)]^{3/2}} \cdot \ln\left|\frac{[2(\rho^2-1)+\rho^2 g+[\rho^4 g^2-4(\rho^2-1)]^{1/2}]}{[2(\rho^2-1)+\rho^2 g-[\rho^4 g^2-4(\rho^2-1)]^{1/2}]} \cdot \frac{[\rho^2 g-[\rho^4 g^2-4(\rho^2-1)]^{1/2}]}{[\rho^2 g+[\rho^4 g^2-4(\rho^2-1)]^{1/2}]}\right|\right] \qquad (6a)$$

$$[\rho^4 g^2 - 4(\rho^2-1)] \ge 0$$

$$\sigma_h^2 = J_1^2\sigma_f^2\lambda_1^2\rho^4\left[\frac{(\rho^2 g+2)}{[4(\rho^2-1)-\rho^4 g^2](\rho^2+\rho^2 g)} + \frac{4}{[4(\rho^2-1)-\rho^4 g^2]^{3/2}} \cdot \left[\tan^{-1}\frac{-2\rho^2-2+\rho^2 g}{[4(\rho^2-1)-\rho^4 g^2]^{1/2}} - \tan^{-1}\frac{\rho g}{[4(\rho^2-1)-\rho^4 g^2]^{1/2}}\right]\right] \qquad (6b)$$

$$[\rho^4 g^2 - 4(\rho^2-1)] \le 0$$

The effects of aspect ratio, ρ, on the variance of the capillary pressure head are shown in Figure 1.

Recall from (40) of part 1 that the effective hydraulic conductivity can be written in the form

$$K_e = E[q_1]/J_1 = K_m\left[1+\frac{1}{2}[\sigma_f^2 - 2\alpha E[fh] + \alpha^2\sigma_h^2] + \frac{E[fj]}{J_1}\right] \qquad (7)$$

The analysis in part 1 showed that $E[fh] = (\beta/J)\sigma_h^2$. Using (2) for the hydraulic conductivity covariance function and following the procedure used in the part 1,

$$E[fj] = E\left[f\,\frac{\partial h}{\partial x_1}\right] = J_1\sigma_f^2\rho^2\left[\frac{2\sigma_h^2}{\lambda_1^2 J_1^2\sigma_f^2\rho^4} - \frac{1}{\rho^2(g+1)}\right] \qquad (8)$$

Substituting $\sigma_h^2\beta/J$ and (8) for $E[fh]$ and $E[fj]$, the effective hydraulic conductivity in the direction perpendicular to the bedding can be expressed as follows:

$$K_e = K_m\left\{1+\frac{1}{2}\left[\sigma_f^2\left(\frac{g-1}{g+1}\right) + \left(\alpha^2 - \frac{2\alpha\beta}{J_1} + \frac{4}{J_1^2\lambda_1^2\rho^2}\right)\sigma_h^2\right]\right\} \qquad (9)$$

If the exponential generalization (see part 1) is used, the effective hydraulic conductivity is

$$K_e = K_G \exp\left\{-\alpha H + \frac{1}{2}\left[\sigma_f^2\left(\frac{g-1}{g+1}\right) + \left(\alpha^2 - \frac{2\alpha\beta}{J_1} + \frac{4}{J_1^2\lambda_1^2\rho^2}\right)\sigma_h^2\right]\right\} \qquad (10)$$

In order to demonstrate the effects of statistical anisotropy on the effective hydraulic conductivity, the result of (10) is illustrated in Figure 2 for several values of the aspect ratio. However, the principal effective hydraulic conductivity in the horizontal direction cannot be evaluated directly in this case because of the unidirectional flow assumption.

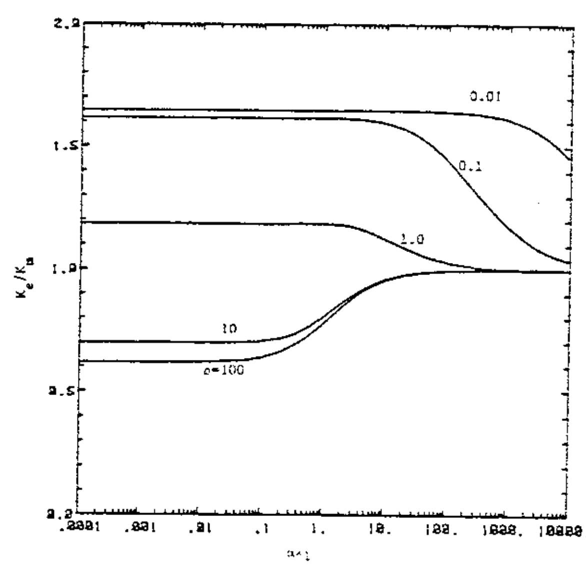

Fig. 2. Effects of aspect ratio, ρ, on the effective hydraulic conductivity K_e, $K_m = K_G \exp(-\alpha H)$, and $\ln K_G = E[\ln K_s]$ for $\sigma_f = 1$ and $J = 1$.

EFFECTIVE HYDRAULIC CONDUCTIVITY OF MULTIDIRECTIONAL FLOW

In this section we investigate the effective hydraulic conductivity in a generalized case where the mean gradient is not restricted to be unidirectional. In other words, the mean gradient vector $\mathbf{J}$ inclines to the axes of the original coordinate system (unprimed system) x_1, x_2, and x_3. In general, the mean gradient vector is composed of three nonzero components: J_1, J_2, and J_3 in the direction of x_1, x_2, and x_3 (see Figure 3), and the mean flow becomes multidimensional. The stratification is aligned in the horizontal axis of the primed coordinate system ($x_3 = x_3'$). The angle between the direction of stratification and the ordinate of the unprimed system, x_2, is θ. The objective is to derive all the components of the effective hydraulic conductivity for this type of porous medium. This analysis is developed for the special case $\lambda/\lambda_1 \to \infty$ in the ln K_s covariance, (2). Again, the parameter α is taken to be constant.

From the relationship of capillary pressure head and hydraulic conductivity Fourier amplitudes (9) of part 1 the head spectrum can be related to the hydraulic conductivity spectrum by

$$S_{hh} = \frac{J_i k_i J_j k_j}{(k^4 + 4\alpha^2 P_i P_j k_i k_j)} S_{ff} \tag{11}$$

where $P_i = J_i - (\delta_{i1}/2)$, δ_{ij} is the Kronecker delta, and the Einstein summation convention is used. Application of the transformation rule $k_i = a_{ni} k_n'$, where $a_{ni} = \cos(x_n', x_i)$, the head spectrum in the primed system takes the form

$$S_{hh}(k') = \frac{\sigma_f^2 J_i J_j a_{mi} a_{nj} k_m' k_n' \lambda_1 \lambda_2 \lambda_3}{\pi^2[k'^4 + 4\alpha^2 P_i P_j a_{mi} a_{nj} k_m' k_n'](1 + k_j'^2 \lambda_j^2)} \tag{12}$$

Integration over the wave number vector $\mathbf{k}$ yields the capillary pressure head variance

$$\sigma_h^2 = \frac{\sigma_f^2 \lambda_1^2 (J_1 a_{11} + J_2 a_{12})^2}{e(1 + e)} \tag{13}$$

where $e = \alpha\lambda_1[(2J_1 - 1)a_{11} + 2J_2 a_{12}]$. Again, the term $E[f\,\partial h/\partial x_i]$ has to be evaluated in order to determine the effective hydraulic conductivity. For this generalized case, the cross-spectrum of f and $\partial h/\partial x_i = j_i$ is

$$\begin{aligned} S_{fji} &= E[-ik_i\, dZ_f\, dZ_h^*] \\ &= \frac{-J_j k_j k_i [k^2 + i(2\alpha J_i k_i - \alpha k_1)] S_{ff}}{[k^4 + (2\alpha J_j k_j - \alpha k_1)^2]} \end{aligned} \tag{14}$$

where $i, j = 1, 2,$ and 3. Using the exponential covariance function (2) and transforming (14) to the primed coordinate system, the generalized spectral relation becomes

$$\begin{aligned} S_{fji}(k') = &-\sigma_f^2 [J_j a_{mi} a_{nj} k_m' k_n' k'^2] \lambda_j \lambda_1 \lambda_2 \lambda_3 \\ &\cdot (\pi^2[k'^4 + 4\alpha^2 P_i P_j a_{mi} a_{nj} k_m' k_n'](1 + k_j'^2 \lambda_j^2)^{-1} \end{aligned} \tag{15}$$

With $\lambda/\lambda_1 \to \infty$ the integration of (15) over wave number space yields [see *Yeh*, 1982]

$$E\left[f\,\frac{\partial h}{\partial x_i}\right] = \frac{-a_{1j} a_{1j} J_j \sigma_f^2}{(1 + e)} \tag{16}$$

Therefore the mean fluxes in the x_1 and x_2 directions can be obtained by substituting (13) and (16) into (12) of part 1. The term $E[jh]$ is evaluated by the relationship provided in (31) of part 1. Consequently, the mean fluxes in the x_1 and x_2 directions in the unprimed coordinate system are

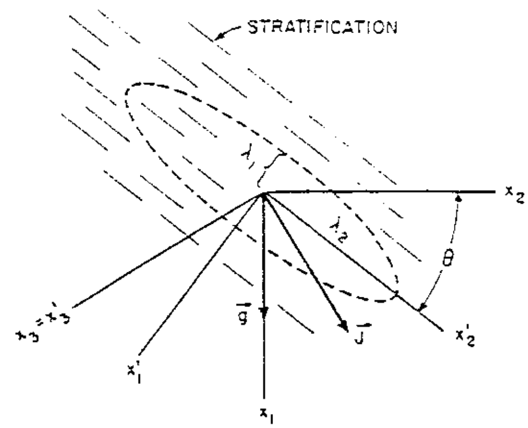

Fig. 3. Coordinate systems for the case of flow in anisotropic ln K_s media with arbitrarily oriented stratification and mean gradient $\mathbf{J}$; $\mathbf{g}$ is the gravitational vector.

$$\begin{aligned} E[q_1] &= K_m\left\{\left[1 + \frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)} - \frac{2a_{11}^2}{1+e}\right)\right]J_1 + \left(\frac{-\sigma_f^2 a_{11} a_{12}}{1+e}\right)J_2\right\} \\ E[q_2] &= K_m\left\{\left(\frac{-\sigma_f^2 a_{11} a_{12}}{1+e}\right)J_1 + \left[1 + \frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)} - \frac{2a_{12}^2}{1+e}\right)\right]J_2\right\} \end{aligned} \tag{17}$$

where $J' = J_1 a_{11} + J_2 a_{12}$, and $N = \alpha\lambda_1$. Equation (17) can be written in the linear form

$$E(q_i) = \bar{K}_{ij} J_j$$

where

$$\bar{K}_{11} = K_m\left[1 + \frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)} - \frac{2a_{11}^2}{1+e}\right)\right] \tag{18a}$$

$$\bar{K}_{22} = K_m\left[1 + \frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)} - \frac{2a_{12}^2}{1+e}\right)\right] \tag{18b}$$

$$\bar{K}_{12} = K_m\left(\frac{-\sigma_f^2 a_{11} a_{12}}{1+e}\right) \tag{18c}$$

$$\bar{K}_{21} = K_m\left(\frac{-\sigma_f^2 a_{11} a_{12}}{1+e}\right) \tag{18d}$$

From the results in (18) it is easily shown that $\bar{K}_{ij}$ transforms as a symmetric tensor of rank two. However, it should be recognized that $\bar{K}_{11}$ and $\bar{K}_{22}$ are functions of J_1 and J_2. The principal axes of $\bar{K}_{ij}$ are found to coincide with the x_1' axes associated with the stratification (Figure 3) and the principal effective hydraulic conductivities becomes

$$\bar{K}_{11}' = K_m\left[1 + \frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)} - \frac{2}{1+e}\right)\right] \tag{19a}$$

$$\bar{K}_{22}' = \bar{K}_{33}' = K_m\left[1 + \frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)}\right)\right] \tag{19b}$$

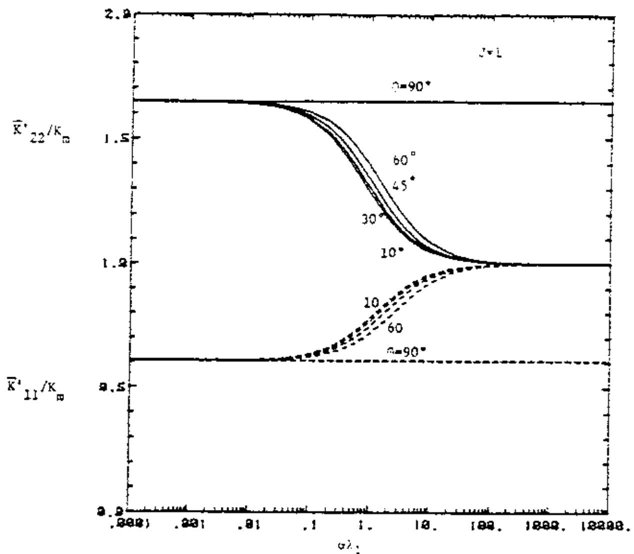

Fig. 4. Principal effective hydraulic conductivities $\bar{K}_{11}'$ (dashed curve) and $\bar{K}_{22}'$ (solid curve) as functions of rotation angle, θ, and $\alpha\lambda_1$, $K_m = K_G \exp(-\alpha H)$, and $\ln K_G = E[\ln K_s]$ for $J_1 = 1$, $J_2 = J_3 = 0$, and $\sigma_f = 1$.

After the exponential generalization (see part 1), the principal hydraulic conductivities take the form

$$\bar{K}_{11}' = K_m \exp\left[\frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)} - \frac{2}{1+e}\right)\right] \tag{20a}$$

$$\bar{K}_{22}' = \bar{K}_{33}' = K_m \exp\left[\frac{\sigma_f^2}{2}\left(1 + \frac{N^2(2a_{11} - 3J')J'}{e(e+1)}\right)\right] \tag{20b}$$

The above hydraulic conductivities yield an anisotropy ratio

$$\frac{\bar{K}_{22}'}{\bar{K}_{11}'} = \exp\left[\frac{\sigma_f^2}{(1+e)}\right] \tag{21}$$

Equation (21) shows that the anisotropy ratio of the hydraulic conductivity depends on not only σ_f^2, $\alpha\lambda_1$, and θ, but also the magnitude and orientation of the mean hydraulic gradient vector **J**. To show the dependence of the principal unsaturated hydraulic conductivities and the anisotropic ratio, normalized conductivities and the anisotropy ratio are plotted as a function of $\alpha\lambda_1$ at different orientations in Figures 4 and 5.

EFFECTS OF RANDOM α PARAMETER

Head Variance

If the parameter α is assumed to be a statistically homogeneous stochastic process, the spectrum of a, S_{aa}, quadspectrum Q_{af}, and cospectrum Co_{af} in the spectral solution of the three-dimensional flow (10) of part 1 are not zero.

In order to evaluate the head spectrum S_{hh}, knowledge of spectra of the a and f processes is necessary. The cross-spectral density functions of the a and f processes have to be known as well. If the cross-covariance of these two processes is known, the cospectrum can be determined. However, no data and information are available for this purpose. It is difficult to assume a reasonable form for the cross-covariance function without any definite justification.

Data on the $K - \psi$ relationship of several different types of soil obtained from *Mualem*'s [1976] soil catalog were used to investigate the relationship between K_s and α. However, no clear relationship is evident. Furthermore, any conclusion on the behavior of such a correlation requires a large amount of field data.

Since no justifiable cospectrum and spectrum of the a process are available, the analysis is carried out with the following assumptions:

1. The autocorrelation functions of the a and f processes are assumed to be equivalent.
2. The a and f processes have the same correlation length scale for the sake of simplicity.
3. The relation of a and f will be considered as either statistically independent or perfectly correlated.

In the following analysis, we will refer to the case where the saturated hydraulic conductivity and pore-size distribution parameter are perfectly correlated as case 1 and where the saturated hydraulic conductivity and pore-size distribution parameter are independent as case 2. If perfect correlation between α and K_s is considered, one can express K_s in terms of α as

$$\zeta \ln K_s = \alpha + \beta \tag{22}$$

where ζ and β are constants; therefore $\zeta f = a$. Making use of the Fourier-Stieltjes representation, the spectrum of the a process is related to f by

$$S_{aa} = \zeta^2 S_{ff} \tag{23}$$

and the cross-spectrum, consequently, becomes

$$S_{af} = \zeta S_{ff} \tag{24}$$

which is real so that $Co_{af} = S_{af}$ and $Q_{af} = 0$. On the other hand, if a and f are statistically independent, (case 2), both the Co_{af} and $Q_{af} = 0$. With these assumptions, the spectral solution (equation (10) of part 1) can be evaluated with the autocovariance function of either the f or a processes.

However, the selection of autocovariance functions or spectra for the saturated hydraulic conductivity and the α parameter fields requires additional consideration. This is due to the

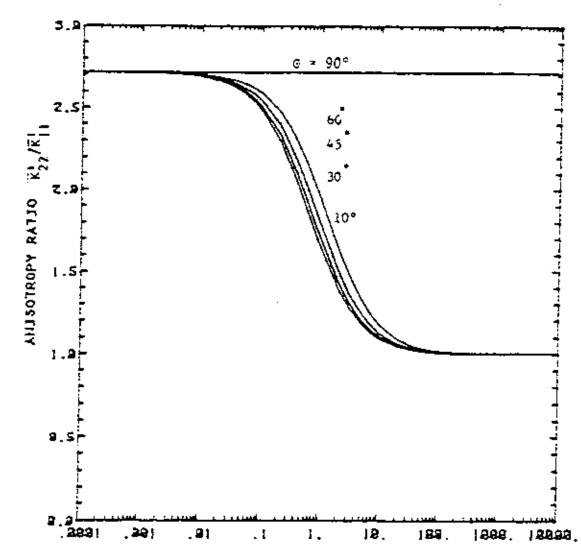

Fig. 5. Ratios of principal hydraulic conductivities as functions of rotation angle θ and $\alpha\lambda_1$ for $\sigma_f = 1$ and $J_1 = 1$.

TABLE 1. Summary of Spectral Relationships, Variances, and Cross-covariances Used in Three-Dimensional Analysis of Flow Through Anisotropic ln K_s and α Random Fields

	Spectral Relationship		Case 1	Case 2
S_{hh}	$\dfrac{k_1^2(S_{ff} - 2HS_{af} + H^2S_{aa})}{(k^4 + A^2k_1^2)}$	σ_h^2	$\dfrac{a_{11}^2\sigma_f^2(1 - H\zeta)^2\lambda_1^2}{e(1 + e)}$	$\dfrac{a_{11}^2(\sigma_f^2 + \sigma_a^2H^2)\lambda_1^2}{e(1 + e)}$
S_{fh}	$\dfrac{k_1^2A(S_{ff} - HS_{af})}{(k^4 + A^2k_1^2)}$	$E[fh]$	$\dfrac{a_{11}^2\sigma_f^2(1 - H\zeta)\lambda_1^2A}{e(1 + e)}$	$\dfrac{a_{11}^2\sigma_f^2A\lambda_1^2}{e(1 + e)}$
S_{ah}	$\dfrac{k_1^2A(S_{af} - HS_{aa})}{(k^4 + A^2k_1^2)}$	$E[ah]$	$\dfrac{a_{11}^2\sigma_f^2(1 - H\zeta)\lambda_1^2A\zeta}{e(1 + e)}$	$\dfrac{-a_{11}^2\sigma_a^2\lambda_1^2AH}{e(1 + e)}$
S_{fj_m}	$\dfrac{-k_1k^2k_m(S_{ff} - HS_{af})}{(k^4 + A^2k_1^2)}$	$E[fj_m]$	$\dfrac{-a_{11}a_{1m}\sigma_f^2(1 - H\zeta)}{(1 + e)}$	$\dfrac{-a_{11}a_{1m}\sigma_f^2}{(1 + e)}$
S_{aj_m}	$\dfrac{-k_1k^2k_m(S_{af} - HS_{aa})}{(k^4 + A^2k_1^2)}$	$E[aj_m]$	$\dfrac{-a_{11}a_{1m}\sigma_f^2(1 - H\zeta)\zeta}{(1 + e)}$	$\dfrac{-a_{11}a_{1m}\sigma_a^2H}{(1 + e)}$

Here, $e = \alpha\lambda_1[(2J_1 - 1)a_{11} + 2J_2a_{12}]$

presence of the last term, $(J_nJ_n - J_1)^2S_{aa}$, in (10) of the part 1. If the three-dimensional anisotropic exponential autocovariance (2) is used for the random process, the resulting head variance is infinite due to singularity of this term at the origin. This mathematical difficulty can be circumvented if a modified exponential autocovariance function [*Naff*, 1978; *Gelhar and Axness*, 1983] is used

$$R_{ff}(\xi) = \sigma_f^2[1 - \xi_1^2/\lambda_1^2s] \exp[-s] \qquad (25)$$

where $s^2 = \xi_1^2/\lambda_1^2 + \xi_2^2/\lambda_2^2 + \xi_3^2/\lambda_3^2$ or the mean gradient is restricted only in the x_1 direction and is assumed to be a unit gradient. The assumption of the unidirectional unit gradient simply eliminates the singularity. In the following analysis, results for the head variance and the effective hydraulic conductivity are obtained with the exponential autocovariance for both a and f processes with the unidirectional, unit gradient assumption.

The resultant head variance is similar to that obtained in the case where the α is a deterministic constant with the exception that the σ_f^2 in (5) is replaced by $\sigma_f^2(1 - \zeta H)^2$ and $(\sigma_f^2 + \sigma_a^2H)$ for cases 1 and 2, respectively. Therefore the head variance for case 1 can be expressed by

$$\sigma_h^2 = J_1^2\sigma_f^2(1 - H\zeta)^2\lambda_1^2\rho^4 \int_0^1 \frac{t^2\,dt}{[(\rho^2 - 1)t^2 + \rho^2gt + 1]} \qquad (26a)$$

Similarly, the head variance for case 2 is given by

$$\sigma_h^2 = J_1^2(\sigma_f^2 + \sigma_a^2H^2)\lambda_1^2\rho^4 \int_0^1 \frac{t^2\,dt}{[(\rho^2 - 1)t^2 + \rho^2gt + 1]} \qquad (26b)$$

Effective Hydraulic Conductivity

To derive the expression for the effective hydraulic conductivity tensor we again assume the hydraulic conductivity at the local scale is homogeneous and isotropic, and the medium is perfectly stratified ($\rho \to \infty$); the generalized effective hydraulic conductivity relationship (12) of part 1 is employed. However, the variance and covariances in (12) are more complicated because of the randomness of α. The spectral or cross-spectral relationships needed to evaluate the variances or covariances in (12) for the case where $J_1 = 1$ and $J_2 - J_3 = 0$ are summarized in Table 1.

Since the mean gradient exists only in the x_1 direction, to obtain a complete description of the hydraulic conductivity tensor we have to examine the case where the stratification of the soil formation is inclined as shown in Figure 3. Thus a coordinate transformation of the spectral relationship as employed in the constant α case is used. After the transformation and the integration over wave number, the variances and covariances for cases 1 and 2 are listed in Table 1.

Substituting the results in Table 1 into (12), the mean flux in the x_1 direction with $J_1 = 1$ becomes

$$E[q_1] = K_m\left\{1 + \frac{\sigma_f^2(1 - H\zeta)^2}{2}\left[\frac{1 - 2a_{11}}{1 + ga_{11}}\right]\right\}J_1 \qquad (27)$$

for case 1, and

$$E[q_1] = K_m\left\{1 + \frac{(\sigma_f^2 + H^2\sigma_a^2)}{2}\left[\frac{1 - 2a_{11}}{1 + ga_{11}}\right]\right\}J_1 \qquad (28)$$

for case 2.

Since the mean gradient exists only in the x_1 direction, i.e., $J_2 = J_3 = 0$, the mean flux equation in the x_2 direction for case 1 becomes

$$E[q_2] = K_mE\left[f\,\frac{\partial h}{\partial x_2}\right] = K_m[\sigma_f^2(1 - H\zeta)^2]\left[\frac{-J_1a_{11}a_{12}}{(1 + ga_{11})}\right] \qquad (29)$$

and

$$E[q_2] = K_mE\left[f\,\frac{\partial h}{\partial x_2}\right] = K_m(\sigma_f^2 + \sigma_a^2H^2)\left[\frac{-J_1a_{11}a_{12}}{(1 + ga_{11})}\right] \qquad (30)$$

for case 2.

At this point it is clear that the difference in the effective hydraulic conductivities of cases 1 and 2 lies in the factor $\sigma_f^2(1 - H\zeta)^2$ and $\sigma_f^2 + \sigma_a^2H^2$. The following analysis will focus on the effective hydraulic conductivity of the medium in which a and f are statistically independent. Therefore the effective hydraulic conductivity tensor components can be written as follows:

$$\bar{K}_{11} = \frac{E[q_1]}{J_1} = K_m\left\{1 + \frac{(\sigma_f^2 + H^2\sigma_a^2)}{2}\left[\frac{1 - 2a_{11}}{1 + ga_{11}}\right]\right\}$$

$$\bar{K}_{21} = \frac{E[q_2]}{J_1} = K_m(\sigma_f^2 + \sigma_a^2 H^2)\left[\frac{-a_{11}a_{12}}{(1 + ga_{11})}\right] \tag{31}$$

Because the effective hydraulic conductivity has a symmetric second-rank tensor property, the hydraulic conductivity in the x_2 direction in the unprimed system, $\bar{K}_{22}$, can be obtained through the relationships between $\bar{K}_{11}$ and $\bar{K}_{21}$ [*Yeh*, 1982]. Finally, the effective hydraulic conductivity tensor $\bar{K}_{ij}$ in the unprimed coordinate system is

$$\bar{K}_{11} = K_m\left[1 + \frac{(\sigma_f^2 + \sigma_a^2H^2)(1 - 2a_{11})}{2(1 + ga_{11})}\right]$$

$$\bar{K}_{22} = \bar{K}_{33} = K_m\left[1 + \frac{(\sigma_f^2 + \sigma_a^2H^2)(1 - 2a_{12})}{2(1 + ga_{11})}\right] \tag{32}$$

$$\bar{K}_{21} = \bar{K}_{12} = K_m(\sigma_f^2 + \sigma_a^2H^2)\left[\frac{-a_{11}a_{12}}{(1 + ga_{11})}\right]$$

The principal effective hydraulic conductivities thus become

$$\bar{K}_{11}' = K_m\left[1 - \frac{(\sigma_f^2 + \sigma_a^2H^2)}{2(1 + ga_{11})}\right]$$

$$\approx K_m \exp\left[\frac{-(\sigma_f^2 + \sigma_a^2H^2)}{2(1 + ga_{11})}\right]$$

$$\bar{K}_{22}' = \bar{K}_{33}' = K_m\left[1 + \frac{(\sigma_f^2 + \sigma_a^2H^2)}{2(1 + ga_{11})}\right] \tag{33}$$

$$\approx K_m \exp\left[\frac{(\sigma_f^2 + \sigma_a^2H^2)}{2(1 + ga_{11})}\right]$$

Note that the effective hydraulic conductivities of media with perfectly correlated a and f can be obtained by replacing $\sigma_f^2 + \sigma_a^2H^2$ with $\sigma_f^2(1 - \zeta H)^2$. The anisotropy ratio of the principal hydraulic conductivities is then

$$\frac{\bar{K}_{22}'}{\bar{K}_{11}'} = \exp\left[\frac{\sigma_f^2(1 - H\zeta)^2}{1 + ga_{11}}\right] \tag{34a}$$

for media with perfectly correlated a and f fields and

$$\frac{\bar{K}_{22}'}{\bar{K}_{11}'} = \exp\left[\frac{(\sigma_f^2 + \sigma_a^2H^2)}{1 + ga_{11}}\right] \tag{34b}$$

for media with statistically independent a and f processes.

Summary and Discussion of Results

Results from the analysis of flow with statistically anisotropic ln K_s and constant parameter α reveal that the head variance resulting from a steady state infiltration in anisotropic media is dependent on the statistical properties of the media, σ_f^2, λ_1, λ_2 and mean gradient J. The effect of the aspect ratio of the medium on the head variance is shown in Figure 1. For flow normal to the bedding of a perfectly stratified soil formation with infinite aspect ratio, the head variance grows infinitely. This result agrees with the result of one-dimensional flow when the exponential covariance function is used. Conversely, head variance vanishes at a zero aspect ratio, which represents the situation where flow is parallel to the bedding of a perfectly stratified soil formation.

Figure 2 illustrates the relationship between the effective unsaturated hydraulic conductivity and the aspect ratio. Generally, the effective unsaturated conductivity follows the well-known behavior of saturated flow in a deterministic medium. The hydraulic conductivity tends to be the arithmetic mean if the medium has a small aspect ratio, which represents the case of flow parallel to the stratification. In a medium with a large aspect ratio representing the case of flow normal to the stratification, the effective hydraulic conductivity becomes less sensitive to the aspect ratio as the correlation scale and pore-size distribution parameter of the medium increase. In this case, it approaches the geometric mean.

The multidirectional flow analysis derives all the components of the unsaturated hydraulic conductivity and shows that unsaturated hydraulic conductivity has the properties of a symmetric tensor of rank two. The effect of $\alpha\lambda$ on the principal unsaturated hydraulic conductivities is depicted in Figure 4. It is found that the effective hydraulic conductivities in the directions parallel and normal to the stratification are arithmetic mean and harmonic mean, respectively. However, they converge to the geometric mean as $\alpha\lambda$ becomes large.

The effect of orientation of the stratification on the anisotropy ratio of unsaturated hydraulic conductivity is illustrated in Figure 5. The orientation effect is significant at intermediate values of $\alpha\lambda_1$, and it is irrelevant at other possible ranges of the parameter. Overall, the degree of anisotropy depends on σ_f^2, $\alpha\lambda_1$, the magnitude and the direction of the gradient, and the orientation of the stratification. A large gradient, α parameter, and correlation scale can reduce the anisotropy ratio. In general, the anisotropy ratio of the unsaturated hydraulic conductivity is smaller than that of saturated hydraulic conductivity. This can be ascribed to the assumption of a constant α and the effect of the nonlinear nature of unsaturated flow. Because the parameter α may vary significantly in the field, the anisotropy ratio derived from the constant α assumption may not represent that anisotropy ratio of the natural soils.

Finally, the head variance and the effective hydraulic conductivity are evaluated for the medium in which ln K_s and α are considered to be stochastic processes. Because α is a stochastic process, knowledge of the cross-covariance between the a and f processes is needed to evaluate the head variance. To simplify the analysis, two cases are considered: case 1, in which α and ln K_s are perfectly correlated, and case 2, in which α and ln K_s are statistically independent. When $J = 1$, both cases yield similar forms of head variance as those derived with the deterministic pore-size distribution parameter assumption. However, the head variance under these circumstances is proportional to $\sigma_f^2(1 - \zeta H)^2$ or $(\sigma_f^2 + \sigma_a^2H^2)$, instead of σ_f^2 as in the constant α case. Thus the head variance can be significantly larger depending on the magnitude of the mean capillary pressure, especially for soils with a large variance of the α parameter, σ_a^2.

The other effect of the variation of α is on the anisotropy ratio of the effective unsaturated hydraulic conductivity. The results in this case reveal that the ratio depends strongly on the mean capillary pressure. The ratio also depends on the magnitude, and direction of the mean gradient, the products of correlation scales, the mean α, and the orientation of the stratification. In other words, the anisotropy of effective unsaturated hydraulic conductivity is dependent on the moisture content and the hydraulic gradient.

The effect of the cross-correlation between the a and f processes on the anisotropy ratio can be elaborated from (34*a*) and (34*b*). As shown by (34*a*), the correlation tends to reduce the anisotropy at low mean capillary pressure. The medium becomes isotropic at the moisture content where H equal to ζ^{-1}. After this critical pressure, the anisotropy ratio increases with capillary pressure. Although the a and f processes in the field may be neither perfectly correlated nor independent, the behavior of field anisotropy will likely fall in between the predictions of (34*a*) and (34*b*).

It is of importance to recognize the limitations and assumptions employed in this study. The theoretical analysis is carried out with the assumption that σ_f^2 and σ_a^2 are small in some sense. The validity of omitting the second order terms in developing the perturbation equations may be questioned, since no exact solutions are available for comparative purposes. However, experience with saturated flow indicates that the perturbation results are quite robust [*Gutjahr and Gelhar*, 1981; *Mizell et al.*, 1982], and the comparisons with Monte Carlo simulations for unsaturated flow by *Anderson and Shapiro* [1983] are quite favorable. Similar questions apply to the exponential generalization of the effective hydraulic conductivity approximations; a more detailed discussion of this aspect is offered in the work by *Gelhar and Axness* [1983]. There is a need for accurate multidimensional Monte Carlo simulation in order to evaluate of these approximations.

The exponential relationship between the conductivity and capillary pressure head assumed in the analysis merely simplifies the mathematics involved. Other functions can be adopted to obtain similar results. In fact, the exponential relationship has been widely used in many practical studies of unsaturated flow. One of the disadvantages is that this relationship may not adequately describe the behavior of the unsaturated conductivity of some soils, especially coarse-textured soils near saturation.

In the development of spectral relationships of capillary pressure head, log-saturated hydraulic conductivity, and the α parameter, stationarity of the head process was assumed in order to use the Fourier-Stieltjes representation theorem. Local stationarity is a reasonable assumption when mean hydraulic gradient is constant but is questionable under flow conditions involving abrupt changes in mean capillary pressure.

The assumption that both the α parameter and log-saturated hydraulic conductivity have the same correlation scale and covariance function is a convenience in the analysis. Similar analyses can be carried out for more complex problems where the statistical properties of the two parameters are not necessarily equal. General behavior of the results should remain similar.

The cross-covariance function of α and $\ln K_s$ may be important in field application of the stochastic results. Analyses have shown that the difference between the results of cases 1 and 2 are significant only at low capillary pressure ranges. The results of these two cases should provide the upper and lower limits of the results of any realistic problem.

Finally, it should be pointed out again that the stochastic results are in the sense of ensemble average or ensemble variance. In order to apply the results of the stochastic analysis to a field situation, it is necessary to invoke the ergodic hypothesis. This hypothesis implies that the scale of the problem under study has to be many times larger than the correlation scale of the input process. Thus an equivalence between ensemble average and space average can be achieved.

The theoretical developments in this paper demonstrate two important phenomena which are a consequence of the variability of the parameter α:

1. The variance of the capillary pressure head increases with increasing mean capillary pressure.
2. The anisotropy of effective unsaturated hydraulic is strongly dependent on the mean capillary pressure.

Observations and applications relating to these features are explored in part 3 of this series.

APPENDIX: EVALUATION OF σ_h^2 IN (6*a*) AND (6*b*)

The variance of the head perturbation can be obtained by integrating the head spectrum (4) over wave number space. To do so, we let $u_i = \lambda_i k_i$ (no sum on i) and assume $\lambda_2 = \lambda_3 = \lambda$. The integral can be written as

$$\sigma_h^2 = \frac{J^2 \sigma_f^2 \lambda_1^2 \rho^4}{\pi^2} \iiint_{-\infty}^{\infty} u_1^2 \, du_1 \, du_2 \, du_3$$

$$\cdot ([(\rho^2 u_1^2 + u_2^2 + u_3^2)^2 + \rho^4 g^2 u_1^2](1 + u^2)^2)^{-1} \qquad \text{(A1)}$$

where $\rho = \lambda/\lambda_1$ and $g = \lambda_1 \beta$. The integral can then be evaluated as follows. First, we can express (A1) in spherical coordinates: $u_1 = u \cos \Phi$, $u_2 = u \cos \Phi \sin \theta$, and $u_3 = u \sin \Phi \cos \theta$, and integrate over θ. The integral becomes

$$\frac{2J^2 \sigma_f^2 \lambda_1^2 \rho^4}{\pi} \int_0^{\pi} \int_0^{\infty} u^2 \cos^2 \Phi \sin \Phi \, d\Phi \, du$$

$$\cdot ([u^2(\sin^2 \Phi + \rho^2 \cos^2 \Phi)^2 + \rho^4 g^2 \cos^2 \Phi](1 + u^2)^2)^{-1} \qquad \text{(A2)}$$

Next, integrating (A2) over u, which can be achieved by the partial fraction technique, yields

$$\sigma_h^2 = J^2 \sigma_f^2 \lambda_1^2 \rho^4 \int_0^1 \frac{t^2 \, dt}{[(\rho^2 - 1)t^2 + \rho^2 g t + 1]^2} \qquad \text{(A3)}$$

where $t = \cos \Phi$.

Finally, the integration of (A3) over t can be obtained by utilizing the formula provided by *Dwight* [1961, section 160.22, p. 39], and it leads to the head variance expressed by (6*a*) and (6*b*).

The technique used here to integrate the head variance can be directly employed to evaluate the integrals related to the variances and covariances in part 3.

REFERENCES

Andersson, J., and A. M. Shapiro, Stochastic analysis of one-dimensional steady state unsaturated flow: A comparison of Monte Carlo and perturbation methods, *Water Resour. Res.*, *19*, 121–133, 1983.

Byer, E., and D. Stephens, Statistical and stochastic analysis of hydraulic conductivity and particle size in a trivial sand, *Soil Sci. Soc. Am. J.*, 47, 1072–1080, 1983.

Dwight, H. B., Tables of integrals and other mathematical data, 336 pp., MacMillan, New York, 1961.

Gelhar, L. W., and C. L. Axness, Three-dimensional stochastic analysis of macrodispersion in aquifers, *Water Resour. Res.*, *19*, 161–180, 1983.

Gutjahr, A. L., and L. W. Gelhar, Stochastic models of subsurface flow: Infinite versus finite domains and stationarity, *Water Resour. Res.*, *17*, 337–350, 1981.

Mizell, S. A., A. L. Gutjahr, and L. W. Gelhar, Stochastic analysis spatial variability in two-dimensional steady groundwater flow assuming stationary and nonstationary heads, *Water Resour. Res.*, *18*, 1053–1067, 1982.

Mualem, Y., A catalogue of the hydraulic properties of unsaturated soils, Development of methods, tools and solutions for unsaturated flow with application to watershed hydrology and other fields, *Res. Proj. 442*, Technion Israel Inst. of Tech., Haifa, 1976.

Naff, R. L., A continuum approach to the study and determination of field longitudinal dispersion coefficients. Ph.D. dissertation, 176 pp., N. M. Inst. of Min. and Tech., Socorro, June 1978.

Russo, D., and E. Bresler, Soil hydraulic properties as stochastic processes, I, An analysis of field spatial variability, *Soil Sci. Soc. Am. J.*, *45*, 699–704, 1981.

Smith, L., A stochastic analysis of steady-state groundwater flow in a bounded domain, Ph.D. thesis, 325 pp., Univ. of B. C., Vancouver, 1978.

Warrick, A. W., G. J. Mullen, and D. R. Nielsen, Predictions of the Soil Water flux based upon field-measured soil-water properties, *Soil Sci. Soc. Am. J.*, *41*, 14–19, 1977.

Yeh, T.-C., Stochastic analysis of effects of spatial variability on unsaturated flow, Ph.D. dissertation, 249 pp., N. M. of Min. and Tech., Socorro, 1982.

Yeh, T.-C., L. W. Gelhar, and A. L. Gutjahr, Stochastic Analysis of unsaturated flow in heterogeneous soil, 1, Statistically isotropic media, *Water Resour. Res.*, this issue.

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Correction to "Stochastic Analysis of Unsaturated Flow in Heterogeneous Soils, 1 and 2," By T.-C. J. Yeh, L. W. Gelhar, and A. L. Gutjahr

In the paper "Stochastic Analysis of Unsaturated Flow in Heterogeneous Soils, 1 and 2" by T. C. Yeh et al. (*Water Resources Research*, *21*(4), 447–456 and 457–464, 1985), an important correction in some equations has been noticed.

The statement right after equation (4*c*) on p. 448, "Then after neglecting the product of a and h ..." should be deleted. Equation (5*a*) ought to be

$$\ln K = F + f - Ah - aH - ah \qquad \ln = \log_e \qquad (5a)$$

and (5*b*) is

$$\ln K_m = E[\ln K] = F - AH - E[ah] \qquad (5b)$$

In addition, equation (11) should be written as

$$q_i = -K\frac{\partial \phi}{\partial x_i} = \exp(F - AH)\left[1 + (f - aH - Ah - ah) + \frac{(f - aH - Ah - ah)^2}{2} + \cdots\right]\left[J_i + \frac{\partial h}{\partial x_i}\right] \qquad (11)$$

Note that the effects of the additional term $E[ah]$ are incorporated in subsequent equations simply by expressing K_m as in the corrected (5*b*). This additional term is nonzero only for the variable α case (part 2), in which case the required $E[ah]$ is given in Table 1. This correction does not affect the anisotropy ratio calculations in part 3 because it is an isotropic effect.

In equation (6*a*), on p. 458, ≥ 0 should be > 0.

In equation (6*b*), on p. 458, ≤ 0 should be < 0. And if $[\rho^4 g^2 - 4(\rho^2 - 1)] = 0$ where $a = \rho^2 g$ and $b = 2(\rho^2 - 1)$.

$$\sigma_h^2 = J_1^2 \sigma_f^2 \lambda_1^2 \rho^4 / [3a(a + b)^3 b^4]$$

In Table 1, on p. 461, the $E[aj_m]$ for case 2 should be positive, and, on the first line, σ_n^2 should be replaced by σ_h^2.

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