Technical Note

One-dimensional, steady vertical flow in a layered soil profile

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Vertical soil water flow is analyzed for steady-state conditions through stratified profiles. All steady-state conditions are considered including upward flow (evaporation) and downward flow (infiltration) for an arbitrary number of layers of different materials. An algorithm is developed utilizing both analytical and numerical techniques, but perhaps more importantly, the framework provides a qualitative assessment somewhat independent of the mathematical description. Three examples are given including comparisons to earlier experimental results.

Key Words: Unsaturated flow, soil water, moisture profiles.

INTRODUCTION

One-dimensional vertical flow is of importance for infiltration as well as for evaporation from shallow water tables. For steady state conditions Darcy's law gives

$$z - z_i = -\int_{h_i}^{h} \frac{\mathrm{d}h}{1 + (q/K)} \qquad (1)$$

where z is elevation, h is the pressure head, $h = h_i$ at z_i, q is the Darcian velocity and $K(h)$ the unsaturated hydraulic conductivity function. The velocity q will be positive for evaporation and negative for infiltration. The h will have units of length and will be negative when suction conditions exist and zero or positive otherwise. Generally, h will be negative for unsaturated conditions, and positive for saturated conditions, although h can be negative when the 'air-entry value' has not been exceeded for saturated conditions. In this analysis, the functional relationship of K to h will vary with position in the profile, but hysteresis will not be a factor as K will be assumed defined for all depths for a steady condition. Any past history would be taken into account by the defined conductivities.

There exists a vast body of literature for infiltration which may be located, for example, in the proceedings of recent conferences[1,2]. For downward flow (steady infiltration) previous studies were made by Tagaki[3], Zaslavsky[4], Bear[5] and Srinilta *et al.*[6]. Generally, they consider two-layered systems and discuss the pressure head development as a function of flow rate and whether a lower permeability layer overlies a less permeable region or vice versa. Srinlita *et al.* present experimental results for several such combinations in laboratory columns.

Other examples of analyses of heterogenous profiles, but primarily for time-dependent cases, are by Raats[7] and Sisson[8]. Zaslavsky and Sinai[9] studied unsaturated, steady flow in sloping, layered soil. Stephens and Heermann[10] investigated the dependency of anisotropy on saturation in stratified sand. Recently, Yeh[11] analyzed a multi-layered system for steady conditions with an interest in testing results against equivalent homogeneous systems.

For evaporation, Gardner and Fireman[12] developed a solution for flow upward from a shallow water table in a uniform soil profile. This was extended to a two-layered system by Willis[13] and experimental results presented. Ripple *et al.*[14] extended Gardner's results to additional relationships for K and demonstrated how soil-water evaporation rates depend on potential evaporation, water table depth and vapor transfer along with the soil parameters. More recently, Warrick[15] extended the Gardner results to even more conductivity relationships.

An objective of the present study is to develop an algorithm valid for all one-dimensional, steady vertical flow conditions through a profile consisting of any number and combination of layers, upward or downward flow and with prescribed head or flux boundary conditions. Assumptions include isothermal and single-phase (liquid) flow based on equation (1). The emphasis will be on analytical expressions, although the numerical evaluation of equation (1) will be performed in one of the examples.

THEORY

Consider, as in Fig. 1, pressure head in a homogeneous profile as a function of elevation for four cases:

Case 1. $q < 0$ and $|K/q| > 1$ (infiltration)
Case 2. $q < 0$ and $|K/q| < 1$ (infiltration)
Case 3. $q < 0$ and $|K/q| = 1$ (infiltration)
Case 4. $q \geqslant 0$ (evaporation or no flow)

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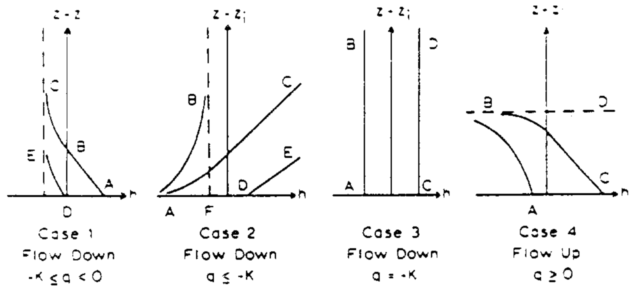

Fig. 1. Possible types of curves for $z - z_i$ vs. h. Case 4 corresponds to equilibrium conditions as well as upward flow

These encompass all possible steady-state cases for both saturated and unsaturated conditions. In the inequalities, K is the unsaturated conductivity and generally varies along the profile. The exact shapes and features of the four cases are quite different. For Case 1, dh/dz is negative (see Fig. 1) and will remain so for the entire layer, regardless of how thick it is. If the layer extends sufficiently upward, the value of h will approach the asymptote (line labelled CE in Fig. 1) defined by $K(h) = -q$. Curves AC and DE are typical h profiles. When h is greater than the air entry value (h_b), K is the saturated conductivity; when h_i is less than h_b, K is less than the saturated value. Also, the profile within the layer is totally defined by equation (1), given the pressure head at the lower elevation of the profile (h_i), the unsaturated hydraulic conductivity function for the layer $K_i(h)$ and the flow q.

For Case 2, dh/dz is positive and h increases with elevation. As before, the slope is constant provided $h > h_b$. If the saturated value of K is greater than $-q$, then an asymptote will be approached (such as Point B of Case 2) otherwise, the pressure head h will increase linearly if the layer extends to sufficiently large values of $z - z_i$ that saturation occurs (cf. AC and DE of Case 2).

For Case 3, the value of dh/dz is zero and consequently, h is constant. For $|q|$ less than the saturated K, profiles similar to AB will result with $h < 0$. If $|q|$ is equal to the saturated K then a profile such as CD will develop with $h \geqslant 0$.

For evaporation (Case 4), q is positive (or zero) and dh/dz will be negative. In this case, a maximum for $z - z_i$ (along BD) will be reached. This corresponds to the water table depth of Gardner[12], which is the limiting value in equation (1) when h approaches $-\infty$. If h is positive at $z = z_i$, then BC could be the result; otherwise AB may be typical or in the case of no flow the slope will be a constant -1.

For a stratified profile, we consider $z = z_0, z, \ldots, z_{n-1}, z_n$, corresponding to pressure heads $h = h_0, h, \ldots, h_{n-1}, h_n$ where

$$K(h) = K_i(h) \qquad z_{i-1} < z < z_i \qquad i = 1, \ldots, n \tag{2}$$

As the profile in any layer is totally defined by q, K and h at the lower elevation, a stratified h profile is equivalent to solving for separate curves (as in Fig. 1) and matching at common points h_i ($i = 1, \ldots, n-1$). For example, consider $K_{1,\text{sat}} > |q| > K_{2,\text{sat}}$ where $K_{1,\text{sat}}$ and $K_{2,\text{sat}}$ are the corresponding saturated conductivities. If

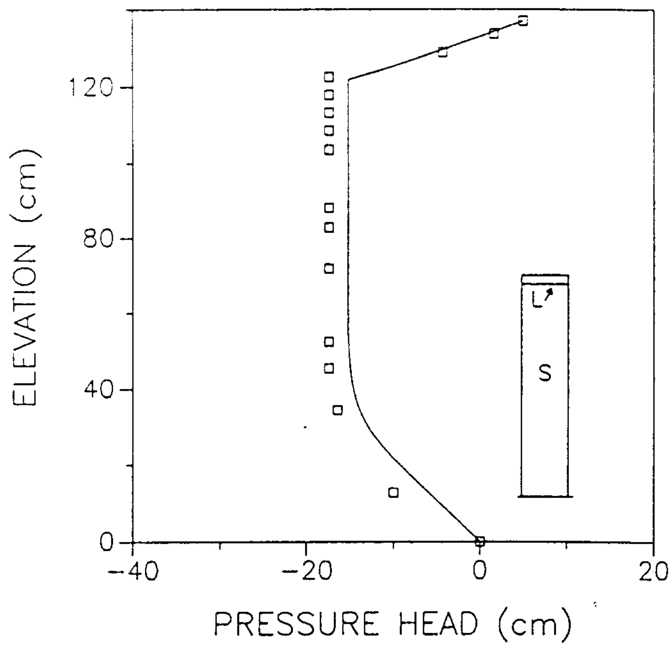

Fig. 2. Profile of h vs. z for Example 1 of 15 cm of Clarion overlying 122 cm of Haegener

$h = 0$ at $z = z_0$, then the resulting composite profile may be similar to part of Curve AC of Case 2 overlying Curve AC of Case 1 from Fig. 1. Such a condition is depicted in Fig. 2 and is equivalent to Fig. 9.4.17 of Bear[8]. The pressure at $z = z_2$ (the uper surface) is totally defined by q, $z = z_0$, $K_1(h)$ and $K_2(h)$. The problem is taken as an 'initial-value' or a 'boundary value' problem in accordance with whether q is given and the resulting h_n calculated or whether h_n is given and q calculated. The two alternatives are fully equivalent, so the choice is arbitrary.

NUMERICAL CALCULATIONS AND EXAMPLES

The necessary steps for calculations are the following:

1. Specify input. These include the number of layers n, the depths defining the layers $z_0, z_1, z_2, \ldots, z_n$, the conductivity functions $K_1(h), K_2(h), \ldots, K_n(h)$, the Darcian velocity q and the lower boundary value h_0.
2. Set $i = 0$.
3. Find h_{i+1} from

$$z_{i+1} - z_i = -\int_{h_i}^{h_{i+1}} \frac{dh}{1 + (q/K)} \tag{3}$$

(Have Case 1 if $K_i(h_i) > -q > 0$;
Case 2 if $-q > K_i(h_i)$; and
Case 3 if $-q = K_i(h_i)$
Case 4 if $q > 0$).

4. Set $i = i + 1$ and repeat Step 3 until h_n has been evaluated.

If h for intralayer points are sought, then use equation (1) with the known values of z_i. If h_n is given rather than q, then use the 'ballistic' method[16] or other techniques.

Some closed forms, relevant to equation (1), are given as Table 1. The forms for K were given by Gardner[12]. He gave the solutions for evaporation ($q > 0$) and for an exponential K and for the second

Table 1. Useful conductitivy functions and integrals

$K(h)$	z + Const.
1. $a \exp(bh)$	$-h - b^{-1} \ln[a + q \exp(-bh)]$
2. $\dfrac{a}{b + (-h)^n}$	$\beta^{-1} \lvert\beta/\alpha\rvert^{1/n} \displaystyle\int \frac{du}{1 \pm u^n}$
	$u = \lvert\alpha/\beta\rvert^{1/n}(-h)$
	$\alpha = q/a \quad \beta = (qb/a) + 1$
	$q\alpha/\beta = \pm\lvert q\alpha/\beta\rvert$
	$q\alpha/\beta > 0$ see Gradshteyn and Ryzhik[18] equation (2.141)–(2.142) for all integer n
	$q\alpha/\beta < 0$ see Gradshteyn and Ryzhik[18] equation (2.143)

form when $n = 1.5$, 2, 3 and 4. Ripple *et al.*[14] presented results for $q > 0$ and integer n for the second form.

The denominator of equations (1) and (3) can approach zero as shown in Fig. 1 along AC of Case 1 or AB of Case 2. For these conditions

$$K(h^* - \delta) = |q| - \delta K' + 0(\delta^2) \tag{4}$$

where $K(h^*) = |q|$, $\delta = h^* - h$ and K' the first derivatives of K evaluated at h^*, signifies a remainder of order δ^2. For these conditions, equation (1) reduces to

$$z - z_i \approx |q/K'|\ln(\delta/\delta_i) \tag{5}$$

with $\delta_i = h^* - h_i$. The error of using equation (5) is of the order $|\delta - \delta_i|$.

Example 1. Clarion sandy clay loam overlying a Hagener sand (Srinilta et al.[6]*)*

Calculations were performed for 15 cm of Clarion sandy loam overlying 122 cm of Hagener sand. The hydraulic functions chosen were after Srinilta *et al.* using the second form of Table 1:

Clarion sandy loam
$K_s = 3.91 \quad b = (4.2)(10)^4 \quad \alpha = 3.36$

Hagener sand
$K_s = 16.7 \quad b = (1.28)(10)^6 \quad \alpha = 5.118$

where K_s is in cm/h and $a = bK_s$.

The resulting profile for $q = -8.9$ cm/h is given in Fig. 2 along with the experimental points[6]. For the lower part (0–120 cm), $|q/K_{sat}| = 8.9/16.7 < 1$ and h decreases with elevation in a manner similar to DE of Case 1 in Fig. 1. Above 120 cm for the Clarion, $|q/K_{sat}| = 8.9/3.9 > 1$ and h increases in a manner similar to AC of Case 2 in Fig. 1.

Example 2. Downward flow through layers of sand and loam

Additional profiles are shown in Fig. 3 using the hydraulic conductivity functions of Willis[13], both of the second form of Table 1 with

$a = (170)(10)^6 \quad b = (2.5)(10)^6 \quad n = 4$ (sand)

and

$a = 700 \quad b = 1450 \quad n = 2$ (Diablo loam)

where h is expressed as cm and the saturated conductivity $K_{sat} = a/b$ is expressed as cm/d. Profiles are plotted for these situations for 10 cm of ponded water on the surface. The curve labelled A is similar to the previous example with the 40 cm of the loam overlying 560 cm of sand. From the top, the h value decreases to the interface, then is nearly constant through the sand (h is approximately -110 cm for which K corresponds to $q = -1.05$ cm/d and finally increases back to $h = 0$ at the base). The second curve is for a four-layered system of 20 cm loam over 280 cm of sand over 20 cm more of loam and finally 280 cm of sand. The resulting q is -1.70 cm/d. The curves for each half of the profile are very similar to that for A with a saturated zone in the middle where the 20 cm of loam serve as an impeding layer. Finally, for the profile labelled C, 40 cm of sand overlies 560 cm of loam. In this case the pressure head h increases from the top to the interface and, in fact, the entire profile is saturated. Below the interface, the pressure head h decreases and becomes zero at the lower boundary. The value of $q = -0.526$, which can be calculated using the well-known composite relationship for an effective K_{sat}[17]:

$$K^*_{sat} = (L_1 + L_2)/[(L_2/K_2) + (L_1/K_1)] \tag{6}$$

For this Case the effective conductivity is $600/[(560/0.483) + (40/68)] = 0.517$. This along with an overall gradient of 610/600 results in $|q| = 0.526$ which is the same as found with the ballistic method.

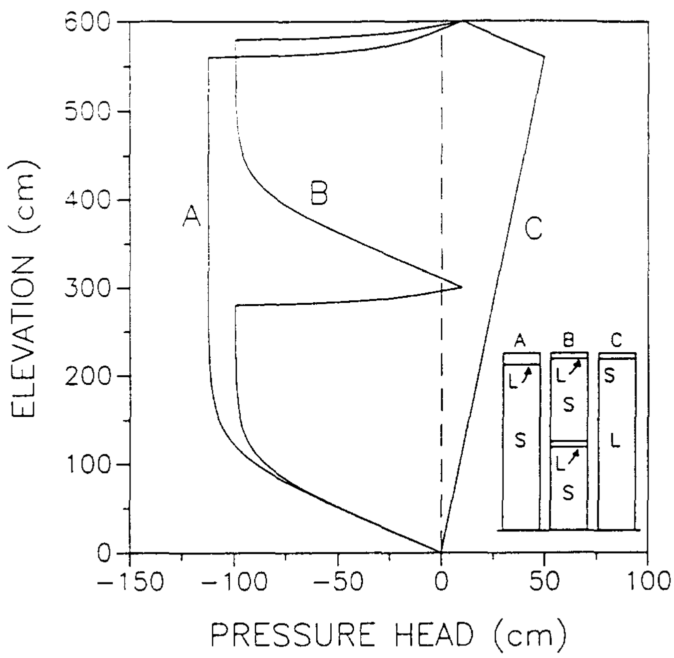

Fig. 3. Profile of h vs. z for Example 2 of loam and sand layers. For Curve A there are 40 cm of loam over 560 cm of sand and q is − 1.05 cm/ d; for Curve B there are 20 cm of loam over 280 cm of sand and repeated with q of − 1.70 cm/ *d; and for Curve C there are 20 cm of sand overlying 560 cm of loam and q is 0.526 cm/ d*

Example 3. Evaporation from layered soils for a shallow water table (Willis[13]*).*

Willis examined a two-layered system of sand over loam and of loam over sand. In both cases, evaporation was measured as a function of water table depth from 50 to a maximum of 268 cm. Hydraulic conductivity functions are the same as for Example 2.

Theoretical profiles are given for a 160 cm profile in Fig. 4. Curve A is for 80 cm of loam over 80 cm of

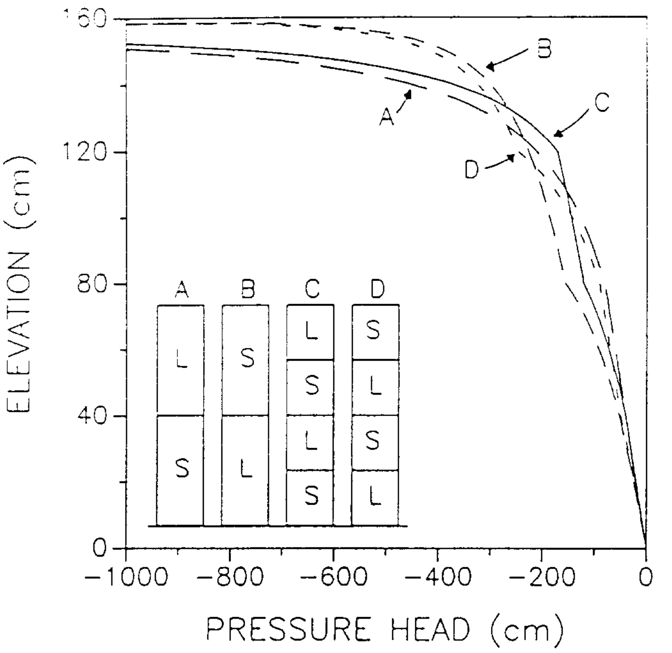

Fig. 4. Profile of h vs. z for evaporation at the soil surface. For descriptions of the four curves see Example 3

sand, Curve B is for 80 cm of sand over 80 cm of loam, Curve C for a four-layered system of 40 cm of loam over 40 cm of sand over 40 cm of loam over 40 cm of sand and Curve D the opposite. Appropriate integrals (see Table 1) are from Gradshteyn and Ryzhik[18] (esp. equations (2.141.2) and (2.141.4)). The results were obtained using the procedure in the previous section by solving as a boundary value problem with the upper surface $z = 160$ cm corresponding to the limit as h approaches negative infinity. As necessary, implicit forms of h as a function of z were solved using a bisection technique[19].

SUMMARY AND CONCLUSIONS

One-dimensional vertical flow has been analyzed based on an overview of typical curves which are defined by direction of flow, the lower boundary conditions and the amount of flow. Analytical expressions can be used for hydraulic conductivity functions when available – otherwise, numerical integrations are appropriate.

Three examples were presented. In Example 1, the experimental results of Srinilta *et al.*[6] were compared with the present solutions. In Example 2, more complex examples were given for up to four layers, although additional layers could have been included with little difficulty.

The advantage of the proposed analysis over existing analytical solutions is that any number of layers may be included and additional conductivity functions may be incorporated. The advantage over purely numerical solutions is that analytical solutions may be incorporated as available. In addition, for flux boundary conditions, this semi-analytical technique always guarantees convergence to the solution. In both cases, the system is examined holistically with the shape within each layer known at the outset. This allows a qualitative delineation of the pressure profile in the heterogeneous soil without total dependence on mathematical models.

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