

Analytical Solutions for One-Dimensional, Transient Infiltration Toward the Water Table in Homogeneous and Layered Soils

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Analytical solutions describing the transient soil water pressure distributions during one-dimensional, vertical infiltration toward the water table through homogeneous and two-layer soils are derived. Exponential functional forms $K = K_s e^{\alpha\psi}$ and $\theta = \theta_r + (\theta_s - \theta_r)e^{\alpha\psi}$ are used to represent the hydraulic conductivity and pressure relation and the soil water release curve. Steady state profiles are used as initial conditions. Hydraulic behavior of the soils during wetting and drainage processes is discussed in terms of the pressure head and moisture content profiles and temporal variation of the specific discharge. The solutions provide a reliable means of comparing the accuracy of various numerical methods, especially in very dry layered soils.

INTRODUCTION

Analytical solutions to the Richards equation for unsaturated flow under various boundary and initial conditions are difficult to obtain because of the nonlinearity in soil hydraulic parameters. This difficulty is exaggerated in the case where soil is heterogeneous. Generally, one has to rely on numerical approaches for predicting moisture movement in unsaturated soils, even for homogeneous soils. However, numerical approaches often suffer from convergence and mass balance problems [Milly, 1985; Celia *et al.*, 1990]. Therefore it is desirable to develop analytical solutions for moisture flow in unsaturated porous media.

During the past few decades, many analytical and quasi-analytical solutions to the unsaturated flow equation have been developed [Philip, 1969]. Most of the solutions were obtained using the exponential hydraulic parameter model proposed by Gardner [1958]. Such an exponential model allows us to linearize the governing flow equation, and analytical solution to the equation thus becomes possible. A detailed review of this approach for unsaturated flow problems was presented by Pullan [1990].

Solutions to the linearized unsaturated flow equations are generally limited to steady state flow in semi-infinite, homogeneous soils. Transient flow analysis is rare and restricted to semi-infinite domains [e.g., Braester, 1973; Warrick, 1973]. Broadbridge and White [1988] presented analytical solutions for transient flow in unsaturated porous media with hydraulic constitutive relations different from the exponential model. But their solutions are limited to the cases with uniform initial conditions and homogeneous or continuous heterogeneous soils. Although a vast body of literature for analytical solutions for steady state vertical infiltration in layered soils is available [e.g., Tagaki, 1960; Zaslavsky, 1964; Srinilta *et al.*, 1969; Raats, 1983; Yeh, 1989; Warrick and Yeh, 1990], analytical solutions for transient vertical infiltration in layered soils are relatively scarce or do not exist.

In this paper we develop analytical solutions to the linear-

ized Richards equations for constant rate vertical infiltration toward the water table in homogeneous soils and two-layer soils. Steady state infiltration soil water pressure profiles are used as initial conditions. Hydraulic behavior of the soils subject to the constant infiltration rate during wetting and drying processes is discussed. Although the solutions based on the exponential functional relations may be very restricted for any practical applications, they do serve as a means for verifying many numerical models for unsaturated flow, especially for infiltration in very dry, layered soils where numerical models often suffer from convergence and mass balance problems. In addition, the analytical solutions may enhance our understanding of the infiltration process under transient state in layered soils. As will be seen in the calculated pressure profiles, even in the linear model the effects of layering are quite interesting.

MATHEMATICAL FORMULATION

The Richards equation governing one-dimensional vertical flow in unsaturated soils can be written as

$$\frac{\partial}{\partial z_*} \left[K_*(\psi) \frac{\partial(\psi + z_*)}{\partial z_*} \right] = \frac{\partial \theta}{\partial t_*} \quad (1)$$

where z_* is the vertical coordinate, positive upward, K_* is the unsaturated hydraulic conductivity which is a function of the pressure head ψ (negative for unsaturated flow), θ is the moisture content, and t_* denotes the time.

In this paper the dependence of the hydraulic conductivity and the moisture content on the pressure head is assumed to be described by the following constitutive relations:

$$K_* = K_s e^{\alpha\psi} \quad (2a)$$

$$\theta = \theta_r + (\theta_s - \theta_r)e^{\alpha\psi} \quad (2b)$$

where K_s is the saturated hydraulic conductivity, θ_r is the residual moisture content, θ_s is the saturated moisture content, and α is a soil pore-size distribution parameter representing the rate of reduction in hydraulic conductivity or moisture content as ψ becomes more negative. It is well known that (2a) and (2b) do not accurately reproduce the

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Paper number 90WR02772.
0043-1327/91/2705-0753\$05.00

soil behavior near saturation. They have been used, nevertheless, in many studies to linearize the governing equation and to obtain an analytic solution. Also, in the following analysis, hysteresis effect is neglected, that is, α is assumed to be the same for wetting and drying cycles.

Using these constitutive relations in (1), a linearized Richards equation can be obtained and expressed as

$$\frac{\partial^2 K_*}{\partial z_*^2} + \alpha \frac{\partial K_*}{\partial z_*} = \frac{\alpha(\theta_s - \theta_r)}{K_s} \frac{\partial K_*}{\partial t_*} \quad (3)$$

A particular solution to this equation requires one initial and two boundary conditions. In this study we examine the case of constant flux at the soil surface, prescribed pressure at the lower boundary, and an initial pressure distribution corresponding to the steady state for a prescribed initial flux at the surface and the prescribed pressure at the lower boundary. The Laplace transformation technique is used for the solution of (3) for two cases: (1) a homogeneous soil and (2) a two-layer soil.

Homogeneous Soil

Consider one-dimensional vertical infiltration toward the water table through a homogeneous soil. L_* is the depth to the water table so that $z_* = 0$ at the water table and $z_* = L_*$ at the land surface; $\psi_0 = 0$ is the prescribed pressure at the water table (all the results in this study are also valid for any other specified pressure ψ_0 at the lower boundary); q_A^* is the initial flux at the soil surface which, along with ψ_0 , determines the initial pressure distribution in the soil; and q_B^* is the prescribed flux at the soil surface for times greater than 0. For convenience, the following dimensionless parameters are defined and used in the rest of this section:

$$z = \alpha z_* \quad \text{so that} \quad L = \alpha L_* \quad (4a)$$

$$K = K_*/K_s \quad (4b)$$

$$q_A = q_A^*/K_s \quad q_B = q_B^*/K_s \quad (4c)$$

$$t = \frac{\alpha K_s t_*}{\theta_s - \theta_r} \quad (4d)$$

Equation (3) can then be written as

$$\frac{\partial^2 K}{\partial z^2} + \frac{\partial K}{\partial z} = \frac{\partial K}{\partial t} \quad (5)$$

The initial and boundary conditions are

$$K(z, 0) = q_A - (q_A - e^{\alpha\psi_0})e^{-z} = K_0(z) \quad (6a)$$

$$K(0, t) = e^{\alpha\psi_0} \quad (6b)$$

$$\left[\frac{\partial K}{\partial z} + K \right]_{z=L} = q_B \quad (6c)$$

After taking the Laplace transform of (5) and denoting the transform of K by \bar{K} , we can write

$$\frac{\partial^2 \bar{K}}{\partial z^2} + \frac{\partial \bar{K}}{\partial z} - s\bar{K} + K_0 = 0 \quad (7)$$

The corresponding boundary conditions (equations (6b) and (6c)) are

$$\bar{K}(0) = e^{\alpha\psi_0/s} \quad (8a)$$

$$\left[\frac{\partial \bar{K}}{\partial z} + \bar{K} \right]_{z=L} = \frac{q_B}{s} \quad (8b)$$

There is a very simple particular solution $K_0(z)/s$ to this special differential equation. The general solution of (7) subjected to the boundary conditions given by (8a) and (8b) is

$$\bar{K} = \frac{K_0(z)}{s} + (q_B - q_A)e^{(L-z)/2}F(s) \quad (9a)$$

where

$$F(s) = \frac{1}{s} \frac{\sinh [z(s + \frac{1}{4})^{1/2}]}{\frac{1}{2} \sinh [L(s + \frac{1}{4})^{1/2}] + (s + \frac{1}{4})^{1/2} \cosh [L(s + \frac{1}{4})^{1/2}]} \quad (9b)$$

The inversion of $F(s)$ is achieved by use of the residue theorem [Özisiç, 1980, p. 278] as the sum of residues of $e^{st}F(s)$ at the poles of $F(s)$. One simple pole is at $s = 0$, and the residue at $s = 0$ is $e^{-(L-z)/2} - e^{-(L+z)/2}$. The other poles are obtained by setting $(s + \frac{1}{4})^{1/2}$ as a complex number. It is easily seen that all the poles are at pure imaginary values of $(s + \frac{1}{4})^{1/2}$, which are expressed as $i\lambda$. The values of λ are obtained as the positive roots (the negative roots are ignored because of symmetry) of the characteristic equation

$$\tan(\lambda L) + 2\lambda = 0 \quad (10)$$

and the residue at λ_n , the n th root of (10), is obtained as

Residue at n th pole

$$= \frac{4 \sin(\lambda_n L) \sin(\lambda_n z) \exp[-(\lambda_n^2 + \frac{1}{4})t]}{1 + (L/2) + 2\lambda_n^2 L} \quad (11)$$

Therefore the expression for K is obtained as

$$K = q_B - (q_B - e^{\alpha\psi_0})e^{-z} - 4(q_B - q_A) \cdot e^{(L-z)/2} e^{-t/4} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n z) \sin(\lambda_n L) e^{-\lambda_n^2 t}}{1 + (L/2) + 2\lambda_n^2 L} \quad (12a)$$

The outflow at the water table ($z = 0$) at any time can be derived as

$$q_t = K_s \left[\frac{\partial K}{\partial z} + K \right]_{z=0} = K_s q_B - 4K_s (q_B - q_A) \cdot e^{L/2} e^{-t/4} \sum_{n=1}^{\infty} \frac{\lambda_n \sin(\lambda_n L) e^{-\lambda_n^2 t}}{1 + (L/2) + 2\lambda_n^2 L} \quad (12b)$$

in which q_t is the outflow in the units of length/time at time t . The equation for K is similar in nature to that obtained by Braester [1973] for a uniform initial moisture content and

finite soil column thickness but is much simpler in appearance. Expressions (12a) and (12b) were found to converge very well for a variety of problems tested, but for some cases it was found that the convergence at small times was very slow. For such cases an alternative expression for K and q_t can be obtained by expanding $F(s)$ as a power series and disregarding the high negative powers of s (because s is very large for small t).

Layered Soil

Now, consider the case where the soil profile consists of two distinct soil layers. The datum (i.e., $z_* = 0$) is assumed to be at the interface of the two layers. In the following analysis, subscript 1 denotes the lower layer and 2 denotes the top layer. Again, dimensionless parameters are defined as

$$z = \alpha_1 z_* \text{ for } -L_{*1} \leq z_* \leq 0 \text{ so that } L_1 = \alpha_1 L_{*1} \quad (13a)$$

$$z = \alpha_2 z_* \text{ for } 0 \leq z_* \leq L_{*2} \text{ so that } L_2 = \alpha_2 L_{*2} \quad (13b)$$

$$K_1 = K_{*1}/K_{s1} \quad q_{A1} = q_A^*/K_{s1} \quad q_{B1} = q_B^*/K_{s1} \quad (13c)$$

$$K_2 = K_{*2}/K_{s2} \quad q_{A2} = q_A^*/K_{s2} \quad q_{B2} = q_B^*/K_{s2} \quad (13d)$$

$$t = \frac{\alpha_1 K_{s1} t_*}{\theta_{s1} - \theta_{r1}} \quad (13e)$$

The governing equations are then written as

$$\frac{\partial^2 K_1}{\partial z^2} + \frac{\partial K_1}{\partial z} = \frac{\partial K_1}{\partial t} \quad (14a)$$

$$\frac{\partial^2 K_2}{\partial z^2} + \frac{\partial K_2}{\partial z} = \beta \frac{\partial K_2}{\partial t} \quad (14b)$$

where

$$\beta = \frac{\alpha_1 K_{s1} (\theta_{s2} - \theta_{r2})}{\alpha_2 K_{s2} (\theta_{s1} - \theta_{r1})} \quad (14c)$$

The initial and boundary conditions along with the interface continuity of flux and pressure are described by the following parameters: The initial condition in layer 1 is

$$K_1(z, 0) = q_{A1} - (q_{A1} - e^{\alpha_1 \psi_0}) e^{-(L_1 + z)} = K_{10}(z) \quad (15a)$$

The initial condition in layer 2, obtained by putting the pressure in layer 2 at the interface equal to the pressure in layer 1, is

$$K_2(z, 0) = q_{A2} - \{q_{A2} - [q_{A1} - (q_{A1} - e^{\alpha_1 \psi_0}) e^{-L_1}] \alpha_2 / \alpha_1\} e^{-z} = K_{20}(z) \quad (15b)$$

The specified pressure at the lower boundary is

$$[K_1]_{z=-L_1} = e^{\alpha_1 \psi_0} \quad (15c)$$

Specified flux at the upper boundary is

$$\left[\frac{\partial K_2}{\partial z} + K_2 \right]_{z=L_2} = q_{B2} \quad (15d)$$

Continuity of flux at the interface is

$$K_{s1} \left[\frac{\partial K_1}{\partial z} + K_1 \right]_{z=0^-} = K_{s2} \left[\frac{\partial K_2}{\partial z} + K_2 \right]_{z=0^+} \quad (15e)$$

Equality of pressure at the interface is

$$\left[\frac{\ln K_1}{\alpha_1} \right]_{z=0^-} = \left[\frac{\ln K_2}{\alpha_2} \right]_{z=0^+} \quad (15f)$$

Taking the Laplace transform of (14a) and (14b) and denoting the transforms by \bar{K}_1 and \bar{K}_2 , we obtain

$$\frac{\partial^2 \bar{K}_1}{\partial z^2} + \frac{\partial \bar{K}_1}{\partial z} - s \bar{K}_1 + K_{10} = 0 \quad (16a)$$

$$\frac{\partial^2 \bar{K}_2}{\partial z^2} + \frac{\partial \bar{K}_2}{\partial z} - \beta s \bar{K}_2 + \beta K_{20} = 0 \quad (16b)$$

Boundary conditions are

$$[\bar{K}_1]_{z=-L_1} = \frac{e^{\alpha_1 \psi_0}}{s} \quad (17a)$$

$$\left[\frac{\partial \bar{K}_2}{\partial z} + \bar{K}_2 \right]_{z=L_2} = \frac{q_{B2}}{s} \quad (17b)$$

with interface conditions

$$\left[\frac{\partial \bar{K}_1}{\partial z} + \bar{K}_1 \right]_{z=0^-} = \frac{K_{s2}}{K_{s1}} \left[\frac{\partial \bar{K}_2}{\partial z} + \bar{K}_2 \right]_{z=0^+} \quad (17c)$$

$$[\bar{K}_1]_{z=0^-} = [LT(K_2^{\alpha_1/\alpha_2})]_{z=0^+} \quad (17d)$$

in which LT indicates Laplace transform. General solutions of (16a) and (16b) can be obtained as

$$\bar{K}_1 = \frac{K_{10}}{s} + [A \cosh(pz) + B \sinh(pz)] e^{-z/2} \quad (18a)$$

$$\bar{K}_2 = \frac{K_{20}}{s} + [C \cosh(qz) + D \sinh(qz)] e^{-z/2} \quad (18b)$$

where

$$p = \left(s + \frac{1}{4} \right)^{1/2} \quad q = \left(\beta s + \frac{1}{4} \right)^{1/2} \quad (18c)$$

and A, B, C, and D are constants which can be determined from the boundary conditions described by (17a)–(17d). After using conditions (17a)–(17c), (18a) and (18b) become

$$\bar{K}_1 = \frac{K_{10}}{s} + A e^{-z/2} \frac{\sinh[p(L_1 + z)]}{\sinh(pL_1)} \quad (19a)$$

$$\begin{aligned} \bar{K}_2 = & \frac{K_{20}}{s} \\ & + \frac{(q_{B2} - q_{A2})e^{(L_2 - z)/2}[-\sinh(qz) + 2q \cosh(qz)]}{2\beta s^2 \sinh(qL_2)} \\ & - \frac{K_{s1}}{K_{s2}} \{Ae^{-z/2}[\sinh(pL_1) + 2p \cosh(pL_1)] \\ & \cdot [\sinh(q(L_2 - z)) + 2q \cosh(q(L_2 - z))] \} \\ & \cdot [4\beta s \sinh(pL_1) \sinh(qL_2)]^{-1} \end{aligned} \quad (19b)$$

The only unknown in (19a) and (19b), A , can be obtained implicitly by using the interface condition (17d). However, a general solution valid for any value of the ratio α_1/α_2 cannot be obtained using this approach. The only case for which a simple solution can be obtained is when $\alpha_1 = \alpha_2$, and therefore, in this study, only soils with the same α values are considered. Although the effect of the contrast in α is important [Yeh, 1989], the above assumption still enables us to analyze the effect of discontinuous K_s on the moisture movement.

Assuming the α values of the soils to be the same, (19a) and (19b) can be reduced to

$$\bar{K}_1 = \frac{K_{10}}{s} - 4(q_{B1} - q_{A1})e^{(L_2 - z)/2}F_1(s) \quad (20a)$$

$$\bar{K}_2 = \frac{K_{20}}{s} - 4(q_{B1} - q_{A1})e^{(L_2 - z)/2}F_2(s) \quad (20b)$$

where

$$F_1(s) = \frac{q \sinh [p(L_1 + z)]}{D_s} \quad (20c)$$

$$\begin{aligned} F_s(s) = & \frac{1}{2D_s} \left\{ \frac{K_{s1}}{K_{s2}} \sinh(qz) \left[\sinh(pL_1) \right. \right. \\ & + 2p \cosh(pL_1) \\ & \left. \left. - \frac{K_{s2}}{K_{s1}} \sinh(pL_1) \right] \right. \\ & \left. + 2q \sinh(pL_1) \cosh(qz) \right\} \end{aligned} \quad (20d)$$

with

$$\begin{aligned} D_s = & s\{-[\sinh(pL_1) + 2p \cosh(pL_1)] \\ & \cdot [\sinh(qL_2) + 2q \cosh(qL_2)] \\ & + \frac{K_{s2}}{K_{s1}} (1 - 4q^2) \sinh(pL_1) \sinh(qL_2)\} \end{aligned} \quad (20e)$$

Equations (20a) and (20b) are inverted using the residue theorem, as described for the homogeneous soil. It is obvious that both equations (20a) and (20b) have poles which are the roots of the equation $D_s = 0$. One simple pole is at $s = 0$, and the other poles are obtained by putting p and q as complex numbers. Again, it is seen that either p or q or both

should be pure imaginary numbers. Three different cases are possible.

Case A: Both p and q Imaginary

So That $p = i\lambda$ and $q = i\mu$

The poles are given by the following equation:

$$\begin{aligned} & [\sin(\lambda L_1) + 2\lambda \cos(\lambda L_1)][\sin(\mu L_2) + 2\mu \cos(\mu L_2)] \\ & - (1 + 4\mu^2) \frac{K_{s2}}{K_{s1}} \sin(\lambda L_1) \sin(\mu L_2) = 0 \end{aligned} \quad (21a)$$

where

$$\mu = \left(\beta \lambda^2 + \frac{\beta - 1}{4} \right)^{1/2} \quad (21b)$$

A trial procedure is used for obtaining the roots of (21a) in which a value of λ is assumed (starting from $\lambda = 0$ and using very small increments), the value of μ is obtained from (21b), and the expression on the left-hand side of (21a) is evaluated. The process is repeated until there is a change of sign in the value of the expression indicating the existence of a pole in that particular range of λ . The root was found to the desired accuracy by using the half-interval search technique.

Getting the residues of $F_1(s)$ and $F_2(s)$ at all these poles (considering only the positive values of λ because of symmetry), we obtain the sum of residues for (20a) and (20b) as

$$RA_1 = \sum_{n=1}^{\infty} \frac{\sin[\lambda_n(L_1 + z)] \exp[-(\frac{1}{4} + \lambda_n^2)t]}{D_n} \quad (22a)$$

$$\begin{aligned} RA_2 = & \sum_{n=1}^{\infty} \{ \sin(\lambda_n L_1) [\sin(\mu_n(L_2 - z)) \\ & + 2\mu_n \cos(\mu_n(L_2 - z))] \exp(-(\frac{1}{4} + \lambda_n^2)t) \} \\ & \cdot \{ D_n [\sin(\mu_n L_2) + 2\mu_n \cos(\mu_n L_2)] \}^{-1} \end{aligned} \quad (22b)$$

in which D_n is given by

$$\begin{aligned} D_n = & \frac{(\frac{1}{4} + \lambda_n^2)}{\mu_n \lambda_n} [A_{ss} \sin(\lambda L_1) \sin(\mu L_2) \\ & + A_{sc} \sin(\lambda L_1) \cos(\mu L_2) \\ & + A_{cs} \cos(\lambda L_1) \sin(\mu L_2) \\ & + A_{cc} \cos(\lambda L_1) \cos(\mu L_2)] \end{aligned} \quad (23a)$$

with

$$A_{ss} = 4\beta \lambda_n \frac{K_{s2}}{K_{s1}} + \lambda_n(L_1 + \beta L_2) \quad (23b)$$

$$A_{sc} = \frac{K_{s2}}{K_{s1}} \frac{\beta \lambda_n L_2}{2\mu_n} (1 + 4\mu_n^2) + 2\lambda_n \mu_n L_1 - \frac{\beta \lambda_n}{2\mu_n} (2 + L_2) \quad (23c)$$

$$A_{cs} = \frac{K_{s2}}{K_{s1}} \frac{L_1}{2} (1 + 4\mu_n^2) + 2\beta \lambda_n^2 L_2 - 1 - \frac{L_1}{2} \quad (23d)$$

$$A_{cc} = -\mu_n(2 + L_1) - \frac{\beta \lambda_n^2}{\mu_n} (2 + L_2) \quad (23e)$$

Case B: p Real and q Imaginary
So That $p = \lambda$ and $q = i\mu$

This case is feasible only when β is more than 1 and p is less than $((\beta - 1)/4\beta)^{1/2}$. The poles are given by the following equation:

$$[\sinh(\lambda L_1) + 2\lambda \cosh(\lambda L_1)][\sin(\mu L_2) + 2\mu \cos(\mu L_2)] - (1 + 4\mu^2) \frac{K_{s2}}{K_{s1}} \sinh(\lambda L_1) \sin(\mu L_2) = 0 \quad (24a)$$

where

$$\mu = \left(\frac{\beta - 1}{4} - \beta \lambda^2 \right)^{1/2} \quad (24b)$$

Getting the residues at all these poles, we obtain the sum of residues for (20a) and (20b) as

$$RB_1 = \sum_{n=1}^{\infty} \frac{\sinh[\lambda_n(L_1 + z)] \exp[-(\frac{1}{4} - \lambda_n^2)t]}{D_n} \quad (25a)$$

$$RB_2 = \sum_{n=1}^{\infty} \{ \sinh(\lambda_n L_1) [\sin(\mu_n(L_2 - z)) + 2\mu_n \cos(\mu_n(L_2 - z))] \exp[-(\frac{1}{4} - \lambda_n^2)t] \cdot \{D_n [\sin(\mu_n L_2) + 2\mu_n \cos(\mu_n L_2)]\}^{-1} \quad (25b)$$

in which D_n is given by

$$D_n = \frac{(\frac{1}{4} - \lambda_n^2)}{\mu_n \lambda_n} [B_{ss} \sinh(\lambda L_1) \sin(\mu L_2) + B_{sc} \sinh(\lambda L_1) \cos(\mu L_2) + B_{cs} \cosh(\lambda L_1) \sin(\mu L_2) + B_{cc} \cosh(\lambda L_1) \cos(\mu L_2)] \quad (26a)$$

with

$$B_{ss} = 4\beta \lambda_n \frac{K_{s2}}{K_{s1}} + \lambda_n(L_1 + \beta L_2) \quad (26b)$$

$$B_{sc} = \frac{K_{s2}}{K_{s1}} \frac{\beta \lambda_n L_2}{2\mu_n} (1 + 4\mu_n^2) + 2\lambda_n \mu_n L_1 - \frac{\beta \lambda_n}{2\mu_n} (2 + L_2) \quad (26c)$$

$$B_{cs} = -\frac{K_{s2}}{K_{s1}} \frac{L_1}{2} (1 + 4\mu_n^2) + 2\beta \lambda_n^2 L_2 + 1 + \frac{L_1}{2} \quad (26d)$$

$$B_{cc} = \mu_n(2 + L_1) - \frac{\beta \lambda_n^2}{\mu_n} (2 + L_2) \quad (26e)$$

Case C: p Imaginary and q Real
So That $p = i\lambda$ and $q = \mu$

This case is feasible only when β is less than 1 and q is less than $((1 - \beta)/4)^{1/2}$. The poles are given by the following equation:

$$[\sin(\lambda L_1) + 2\lambda \cos(\lambda L_1)][\sinh(\mu L_2) + 2\mu \cosh(\mu L_2)] - (1 - 4\mu^2) \frac{K_{s2}}{K_{s1}} \sin(\lambda L_1) \sinh(\mu L_2) = 0 \quad (27a)$$

where

$$\lambda = \left(\frac{1 - \beta}{4\beta} - \frac{\mu^2}{\beta} \right)^{1/2} \quad (27b)$$

Getting the residues at all these poles, we obtain the sum of residues for (20a) and (20b) as

$$RC_1 = \sum_{n=1}^{\infty} \frac{\sin[\lambda_n(L_1 + z)] \exp[-(\frac{1}{4} + \lambda_n^2)t]}{D_n} \quad (28a)$$

$$RC_2 = \sum_{n=1}^{\infty} \{ \sin(\lambda_n L_1) [\sinh(\mu_n(L_2 - z)) + 2\mu_n \cosh(\mu_n(L_2 - z))] \exp[-(\frac{1}{4} + \lambda_n^2)t] \cdot \{D_n [\sinh(\mu_n L_2) + 2\mu_n \cosh(\mu_n L_2)]\}^{-1} \quad (28b)$$

in which D_n is given by

$$D_n = \frac{(\frac{1}{4} + \lambda_n^2)}{\mu_n \lambda_n} [C_{ss} \sin(\lambda L_1) \sinh(\mu L_2) + C_{sc} \sin(\lambda L_1) \cosh(\mu L_2) + C_{cs} \cos(\lambda L_1) \sinh(\mu L_2) + C_{cc} \cos(\lambda L_1) \cosh(\mu L_2)] \quad (29a)$$

with

$$C_{ss} = 4\beta \lambda_n \frac{K_{s2}}{K_{s1}} + \lambda_n(L_1 + \beta L_2) \quad (29b)$$

$$C_{sc} = -\frac{K_{s2}}{K_{s1}} \frac{\beta \lambda_n L_2}{2\mu_n} (1 - 4\mu_n^2) + 2\lambda_n \mu_n L_1 + \frac{\beta \lambda_n}{2\mu_n} (2 + L_2) \quad (29c)$$

$$C_{cs} = \frac{K_{s2}}{K_{s1}} \frac{L_1}{2} (1 - 4\mu_n^2) + 2\beta \lambda_n^2 L_2 - 1 - \frac{L_1}{2} \quad (29d)$$

$$C_{cc} = -\mu_n(2 + L_1) + \frac{\beta \lambda_n^2}{\mu_n} (2 + L_2) \quad (29e)$$

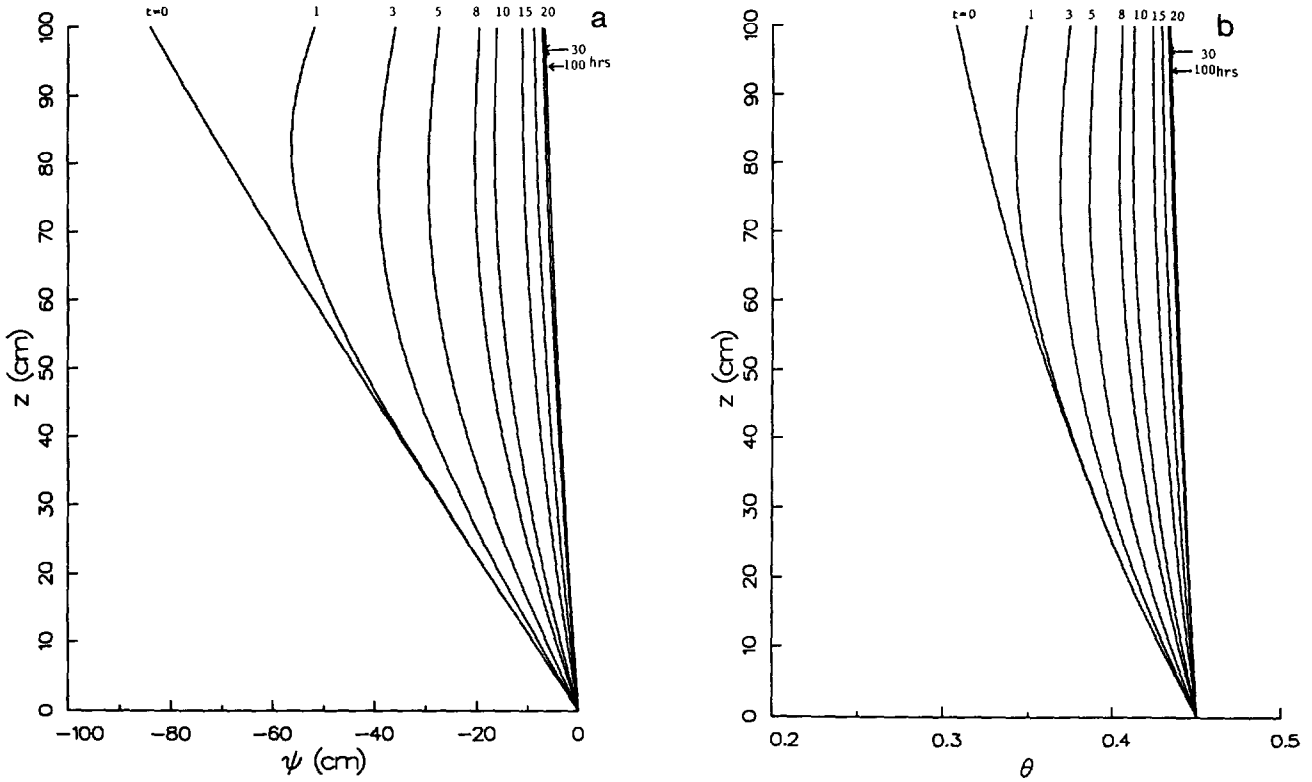


Fig. 1. Wetting profiles for homogeneous soil with $\alpha = 0.01/\text{cm}$: (a) pressure and (b) moisture.

After getting the sum of residues for case A and the relevant case B or C, we can add the residue at $s = 0$ and get the expressions for K_1 and K_2 as

$$K_1 = q_{B1} - (q_{B1} - e^{\alpha\psi_0})e^{-(L_1+z)} - 4(q_{B1} - q_{A1})e^{(L_2-z)/2}[RA_1 + (RB_1 \text{ or } RC_1)] \quad (30a)$$

$$K_2 = q_{B2} - [q_{B2} - q_{B1} + (q_{B1} - e^{\alpha\psi_0})e^{-L_1}]e^{-z} - 4(q_{B1} - q_{A1})e^{(L_2-z)/2}[RA_2 + (RB_2 \text{ or } RC_2)] \quad (30b)$$

The outflow at the water table ($z = -L_1$) at any time is given by

$$q_t = q_B + 4(q_B - q_A) \exp \frac{L_1 + L_2}{2} \cdot \sum_{n=1}^{\infty} \left[\frac{\lambda_n \exp [-(\frac{1}{4} \pm \lambda_n^2)t]}{\Sigma_{A,(B \text{ or } C)} D_n} \right] \quad (30c)$$

EXAMPLES AND DISCUSSION

The analytical expressions (12) and (30), along with the constitutive relations (2a) and (2b), are used in the following two examples to compute the pressure head, moisture content, and specific discharge. These examples include infiltration toward the water table through hypothetical soils (homogeneous and two-layer soils). The resulting pressure head profiles, moisture profiles, and outflow variations are discussed.

Example 1: Homogeneous Soil

In this example the thickness of the homogeneous soil is taken to be 100 cm. The saturated hydraulic conductivity value is assumed to be 1.0 cm/hr. Two α values (0.1/cm and

0.01/cm) are used to illustrate the effect of α on the movement of moisture. The saturated and residual water contents of the soil, with $\alpha = 0.1$, are taken as 0.40 and 0.06, respectively; for $\alpha = 0.01$ they are 0.45 and 0.2.

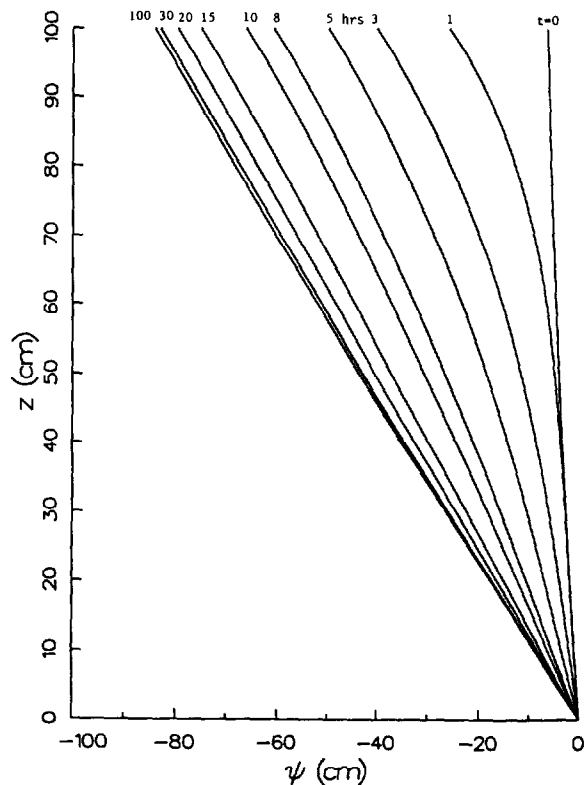


Fig. 2. Pressure profile during drainage for homogeneous soil with $\alpha = 0.01/\text{cm}$.

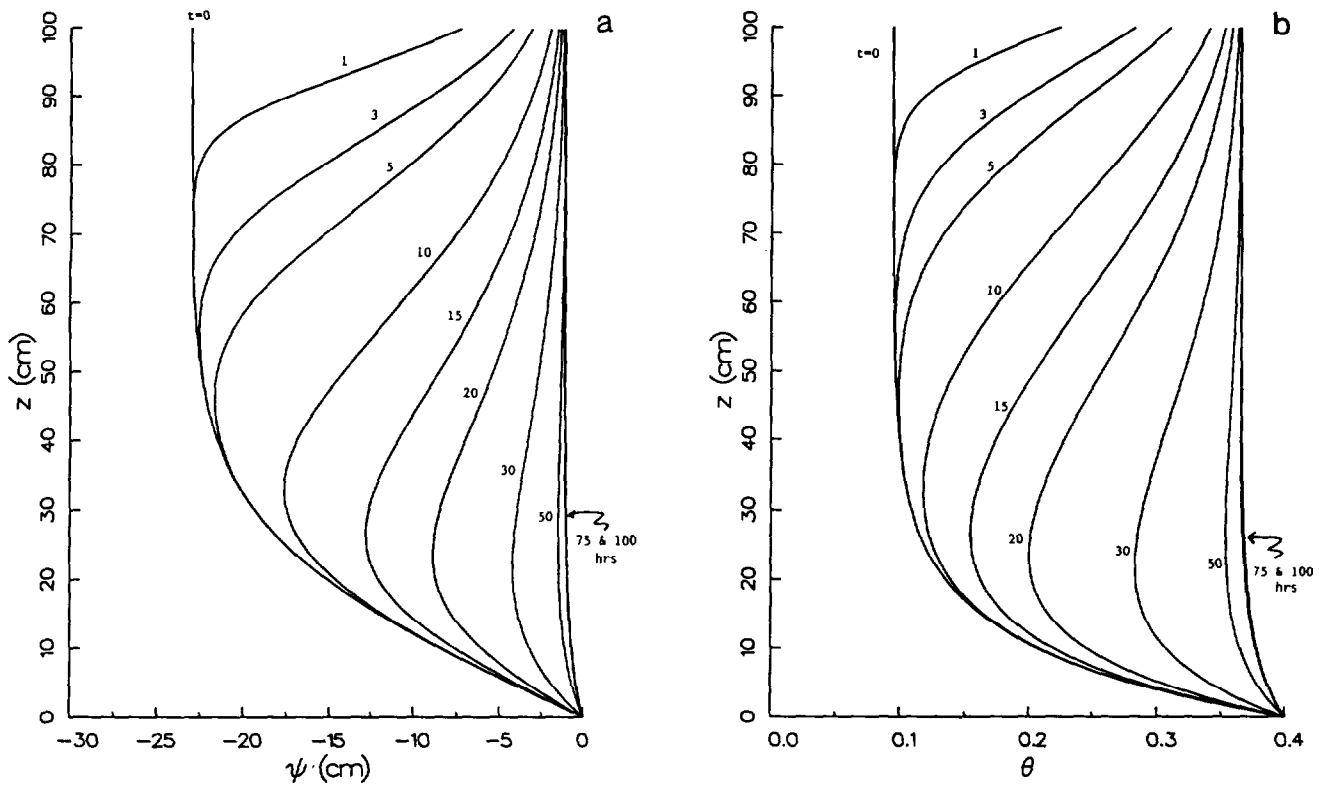


Fig. 3. Wetting profiles for homogeneous soil with $\alpha = 0.1/\text{cm}$: (a) pressure and (b) moisture.

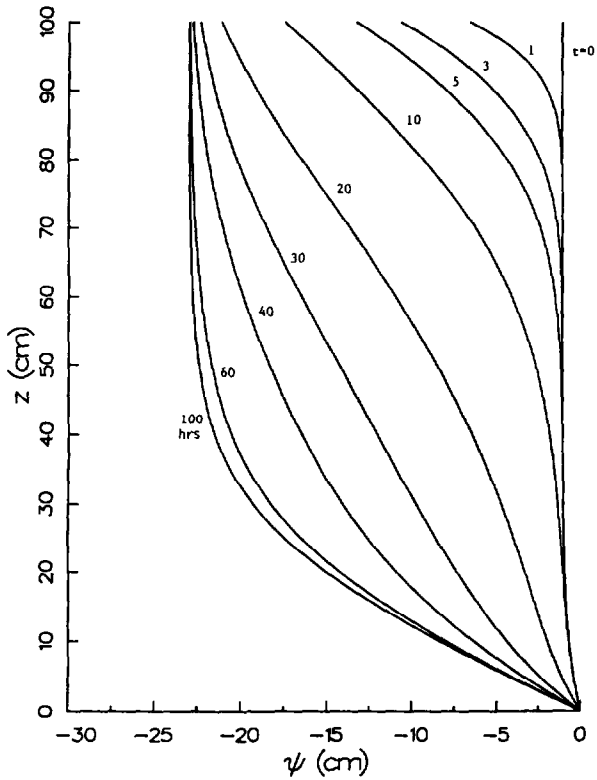


Fig. 4. Pressure profile during drainage for homogeneous soil with $\alpha = 0.1/\text{cm}$.

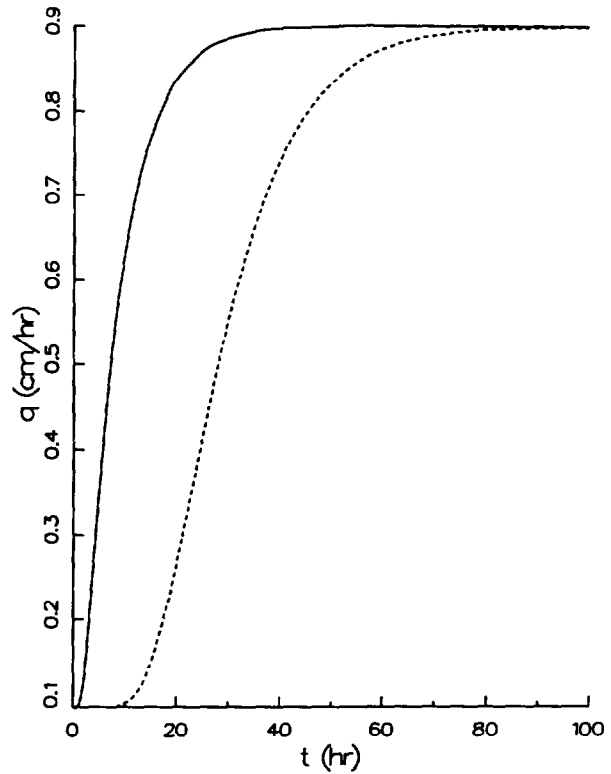


Fig. 5. Outflow at the water table for soils of Figures 1 (solid curve) and 3 (dashed curve).

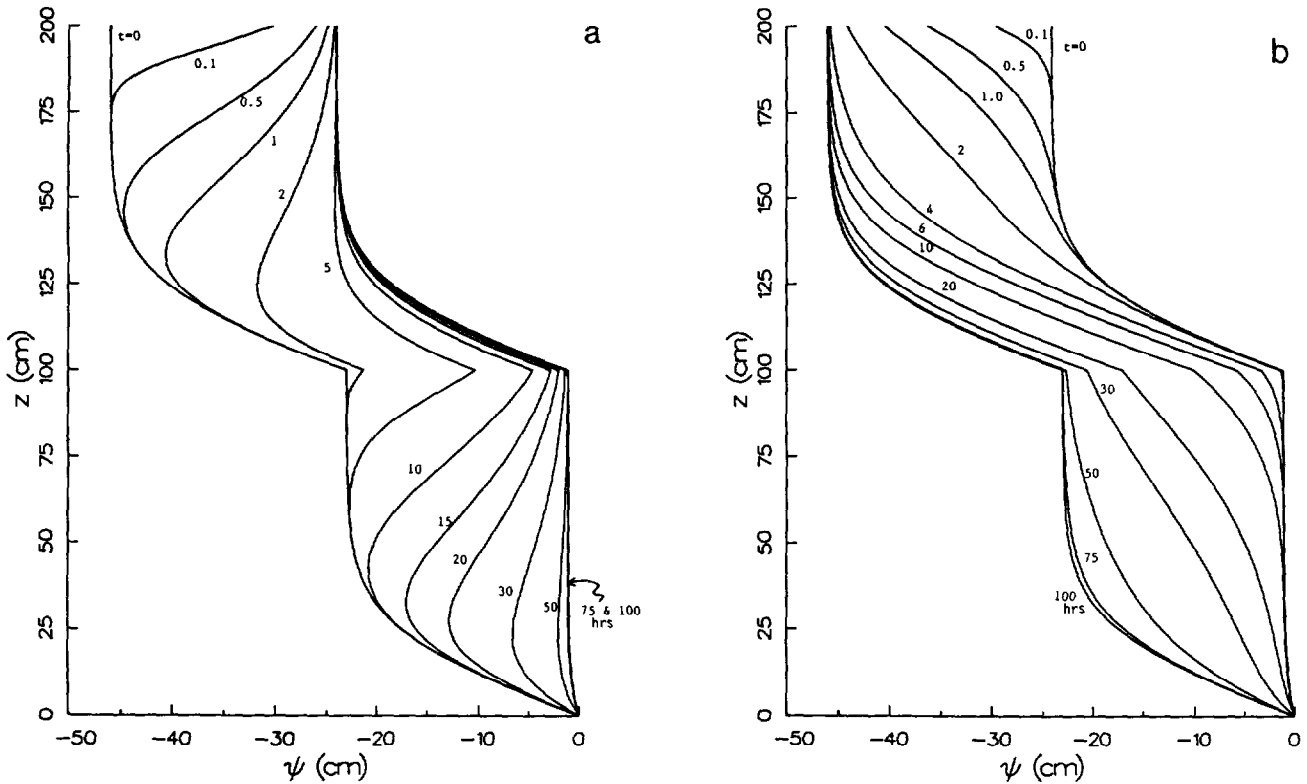


Fig. 6. Pressure profiles for layered soil with $\alpha = 0.1/\text{cm}$, $K_{s1} = 1 \text{ cm/h}$, and $K_{s2} = 10 \text{ cm/h}$: (a) wetting and (b) drainage.

Figure 1a shows the calculated pressure head profiles at several times for a soil with $\alpha = 0.01/\text{cm}$ during a wetting scenario. The initial pressure head profile is the steady state infiltration profile at an infiltration rate equal to 0.1 cm/h. Then, at t greater than 0, the infiltration rate is increased to 0.9 cm/h. The corresponding moisture profiles are shown in Figure 1b. It can be seen that both the moisture profiles and the pressure profiles are similar in shape, which is a result of assuming exponential constitutive relations (2a) and (2b).

The pressure head profiles for the same soil during a drainage scenario, where the infiltration rate is suddenly reduced from 0.9 to 0.1 cm/h, are shown in Figure 2.

The pressure and moisture profiles during the wetting scenario in a soil with the same hydraulic properties as in Figures 1 and 2, except that $\alpha = 0.1/\text{cm}$, are presented in Figures 3a and 3b, respectively. The pressure profile for the drainage scenario is shown in Figure 4. Because of similar shapes of the moisture and pressure profiles, only the pressure profiles are shown in the rest of this section.

A comparison of these figures shows that in the soil with smaller α the wetting and the drying fronts are dispersed rapidly to a large distance, while in the soil with larger α value the wetting and the drying fronts tend to be very sharp. The relevant dimensionless parameter here for determining the profile shape is the (inverse Peclet) number αL .

Discharge as a function of time from the two soils to the water table is plotted in Figure 5. As illustrated in the figure, the increase in discharge from the soil with a small α tends to be earlier, and the flow reaches a new steady state faster than the one from the soil with a large α value.

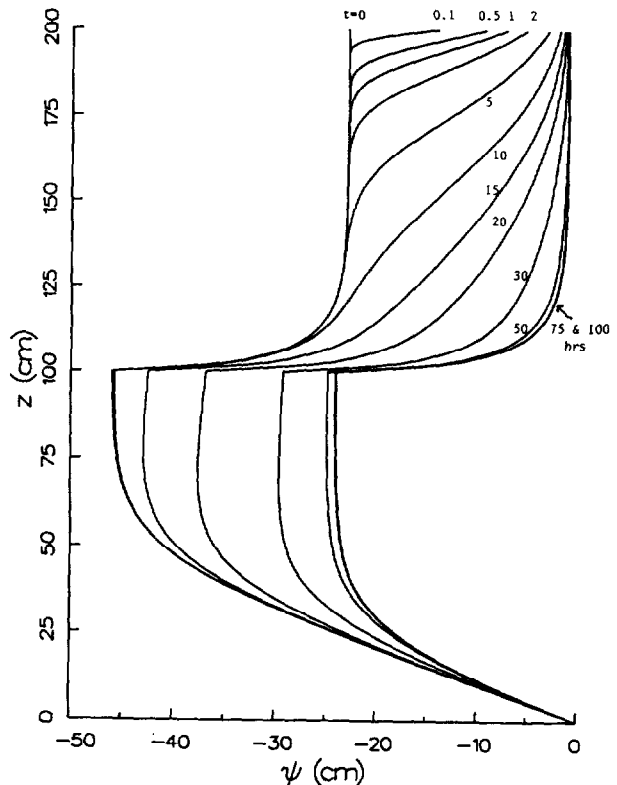


Fig. 7. Pressure profile for the wetting case for a layered soil with $\alpha = 0.1/\text{cm}$, $K_{s1} = 10 \text{ cm/h}$, and $K_{s2} = 1 \text{ cm/h}$.

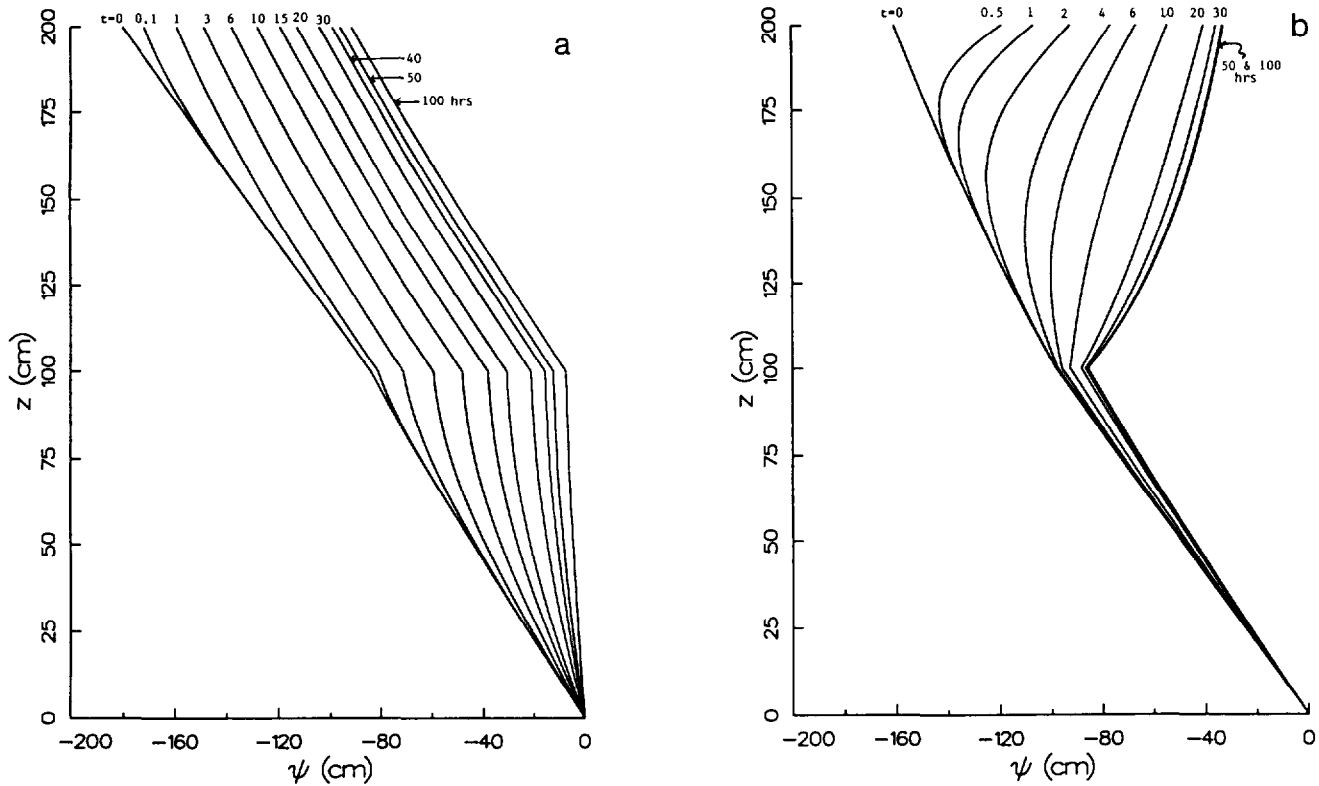


Fig. 8. Pressure profiles for the wetting case for layered soil with $\alpha = 0.01/\text{cm}$: (a) $K_{s1} = 1 \text{ cm/h}$ and $K_{s2} = 10 \text{ cm/h}$ and (b) $K_{s1} = 10 \text{ cm/h}$ and $K_{s2} = 1 \text{ cm/h}$.

Example 2: Two-Layer Soil

Equations (30a) and (30b) are used in this example to calculate the pressure head profile for the wetting and

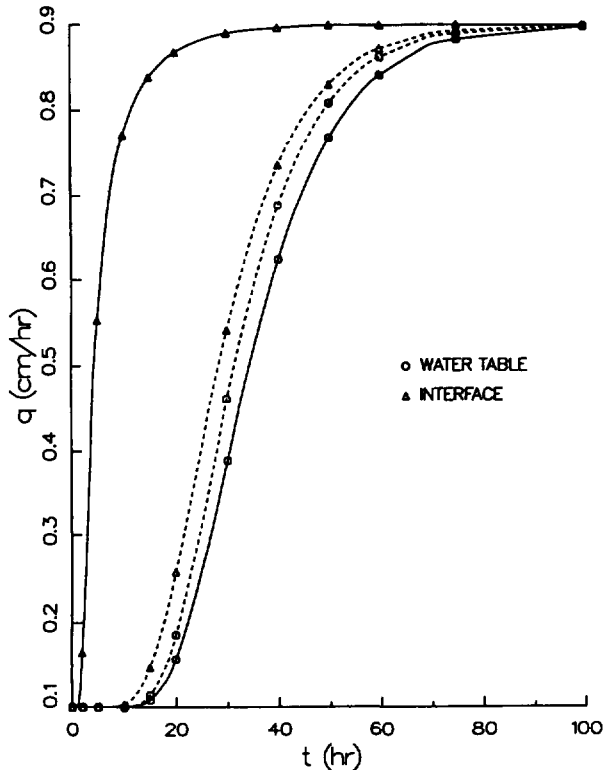


Fig. 9. Specific discharge at the interface and at the water table for Figures 6a (solid curve) and 7 (dashed curve).

drainage scenarios in a two-layered soil. The α of the two layers are set to be $0.1/\text{cm}$, and the saturated hydraulic conductivities of the lower and upper layers are equal to 1 and 10 cm/h, respectively. The thickness of each layer is 100 cm. The saturated and residual water contents are taken as 0.40 and 0.06, respectively. The boundary and initial conditions are the same as those in the homogeneous cases. Figures 6a and 6b show the pressure profiles at different times during the wetting and the drainage scenarios.

Figure 7 depicts the pressure profile at various times during a wetting scenario in a two-layer soil where the top layer has a saturated hydraulic conductivity value equal to 1 cm/h and the bottom layer, 10 cm/h. The boundary and initial conditions are the same as those in Figure 6a.

A comparison of the profiles of Figures 6a and 7 demonstrates the effect of the position of a highly conductive layer on the propagation of the wetting front. In Figures 6a and 6b, infiltration starts in the high conductive layer. Once the wetting front reaches the top of the less conductive layer, the pressure head at the interface increases rapidly so that moisture is able to move through the less conductive layer at the bottom. On the other hand, in Figure 7, infiltration starts in the less conductive layer. Once the wetting front reaches the interface with the high conductive layer, the amount of moisture flow to that layer can be easily dissipated because of the high conductivity of the layer. Thus the hydraulic gradient remains approximately the same (i.e., unit gradient). The flow in the layer behaves as successive steady state flows.

Wetting profiles in layered soils with the same conditions as in Figures 6a and 7, except for a small α value ($0.01/\text{cm}$), are presented in Figures 8a and 8b, respectively. Since the

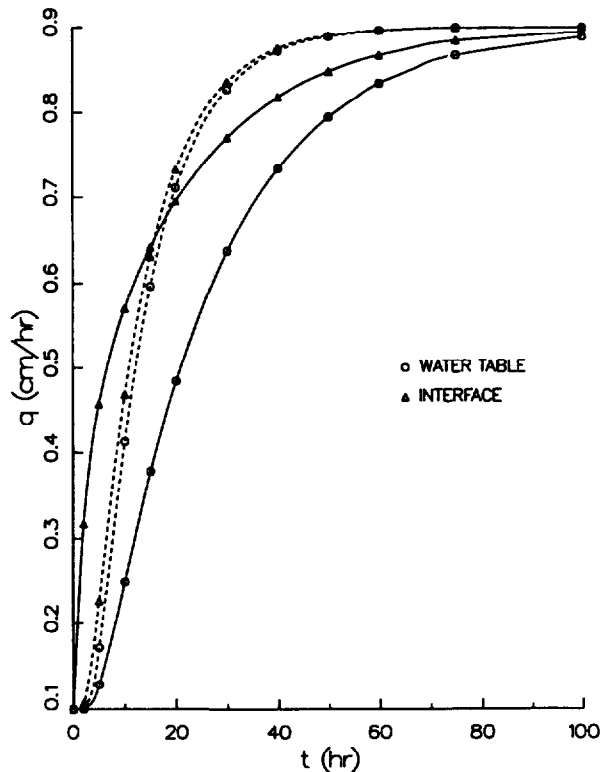


Fig. 10. Specific discharge at the interface and at the water table for Figures 8a (solid curve) and 8b (dashed curve).

value of α is relatively small, the wetting fronts are more dispersed than those in Figures 6a and 7.

Figure 9 shows the specific discharge values at the interface and at the water table, corresponding to Figures 6a and 7. The solid curve denotes the discharge corresponding to Figure 6a and the dashed curve denotes the discharge in Figure 7. Comparing the solid and dashed curves, it can be seen that, as expected, the time lag between the arrival of flow at the interface and at the lower boundary is considerably reduced in Figure 7 because of the higher conductivity of the lower layer. Similar plots for the cases corresponding to Figures 8a and 8b, where $\alpha = 0.01/\text{cm}$, are shown in Figure 10. A comparison of Figures 9 and 10 shows that the

time lag between the flow at the interface and the outflow is considerably smaller for the smaller α .

Acknowledgments. This study is partially funded by EPA grant R-813899-01-1. The support of the University of Arizona is acknowledged, and comments and suggestions of Phil Broadbridge are greatly appreciated.

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(Received September 18, 1990;
revised December 7, 1990;
accepted December 18, 1990.)