Stochastic Analysis of Unsaturated Flow in Heterogeneous Soils
1.   Statistically Isotropic Media

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Steady unsaturated flow with vertical mean infiltration through unbounded heterogeneous porous media is analyzed using a perturbation approximation of the stochastic flow equation which is solved by spectral representation techniques. The hydraulic conductivity K is related to the capillary pressure head ψ by $K = K_s \exp(-\alpha\psi)$, where K_s is the saturated conductivity, and α is a soil parameter. A general formulation is presented for the case with K_s and α represented as statistically homogeneous spatial random fields. In part 1, solutions are developed assuming α is constant and representing K_s variability by one-dimensional and three-dimensional isotropic random fields. Results are obtained for head variances and covariance functions, effective hydraulic conductivities, variances of the unsaturated hydraulic conductivity, flux variances, and variance of pressure gradient. When the parameter α is relatively large, corresponding to coarse textured soils, the head variance decreases and all of the results demonstrate a trend toward gravitationally dominated one-dimensional vertical flow. The effective conductivity is dependent on the correlation scale of $\ln K_s$ and the mean hydraulic gradient.

Introduction

Field observations show that the hydraulic properties of soils vary significantly with spatial location even within a given soil type [*Warrick and Nielsen*, 1980]. For example, the standard deviation of the natural logarithm of saturated hydraulic conductivity has been observed to range from 0.3 to 3. Quantitative information on the behavior of unsaturated flows is based largely on laboratory experiments with small homogenized soil columns or on small-scale agriculturally oriented field experiments in the top few meters of soil. Such observations have established that unsaturated flow is adequately described at those small scales by classical continuum partial differential equations. However, important problems of resource management and environmental protection, such as groundwater recharge prediction or waste disposal evaluation, require quantification at much larger scales, e.g., hundreds of meters or more. A key question is how to extrapolate the classical small-scale homogeneous behavior to the pertinent field scale at which the soil parameters exhibit complex natural heterogeneity.

The standard approach to such field-scale problems has been to construct analytical or numerical solutions of the classical partial differential equations, assuming that the parameters take on known, constant values within zones in the model. Although there have been major advances in numerical solution techniques during the last decade, the standard modeling approach is severly limited by the lack of methods for determining the effective large-scale parameters. Current practice is characterized by numerical overkill with practically nonexistent data. *Nielsen and Biggar* [1982] discuss many limitations of the standard modeling approach in the vadose zone emphasizing the need for statistical techniques which incorporate the effects of natural variability.

Statistical techniques of treating natural variability have

been applied to unsaturated flow [e.g., *Warrick*, 1977; *Eagleson*, 1978; *Dagan and Bresler* 1979; *Amoozegar-Fard et al.*, 1982; *Cordova and Bras*, 1982; *Bresler and Dagan*, 1983], but all of these analyses presume one-dimensional vertical flow in which the parameters are random but spatially constant in the vertical and statistically independent in the horizontal. This conceptual framework is convenient for analysis, but observations [e.g., *Russo and Bresler*, 1980; *Viera et al.*, 1981; *Sisson and Wierenga*, 1981] indicate that saturated hydraulic conductivity variations are correlated over horizontal distances of tens of meters. Also soil profiles are, of course, not vertically homogeneous; recent data reported by *Byer and Stephens* [1983] illustrated the three-dimensional anisotropic character of soil heterogeneity.

For saturated flow, stochastic analyses have clearly demonstrated the importance of the multidimensional spatial correlation structure [e.g., *Gelhar*, 1976; *Bakr et al.*, 1978; *Smith and Freeze*, 1979; *Dagan*, 1982; *Gelhar and Axness*, 1983]. These approaches are based on solutions of the stochastic partial differential equations which result when the hydraulic conductivity is taken as a spatial stochastic process or random field. *Andersson and Shapiro* [1983] present a stochastic solution of this type for one-dimensional steady vertical infiltration. *Philip* [1980] presents a random walk approach to unsaturated flow and emphasizes the importance of the three-dimensionality of field heterogeneity. *Sposito* [1978] explores a different kind of statistical theory which does not address the effects of spatial variability. In this series of papers we treat steady unsaturated flow of three-dimensionally heterogeneous media. The overall goals are to determine the mean or effective large-scale behavior of the heterogeneous system and the degree of variation about the mean. Following *Gardner* [1958], the hydraulic conductivity K is related to the capillary pressure head ψ by

$$K(\psi, x_i) = K_s(x_i) \exp(-\alpha(x_i)\psi) \tag{1}$$

where K_s is the saturated hydraulic conductivity, and α is a parameter which represents the relative rate of decrease of hydraulic conductivity with increasing capillary pressure head. The parameters K_s and α are measurable for small laboratory samples; in the field they are taken to be functions of the spatial coordinates x_i. In this analysis K_s and α are repre-

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sented by three-dimensional statistically homogeneous (stationary) random fields. The capillary pressure head is assumed to be made up of a zero mean perturbation represented by a statistically homogeneous random field and a mean which varies slowly relative to the correlation scales associated with K_s and α. A first-order perturbation approximation is used in developing analytical solutions via spectral representation techniques.

The goal is generic local relationships expressing the ψ variations in terms of the mean capillary pressure head field which will yield the mean equation describing the large-scale behavior. This approach applies only when the scale of heterogeneity is much smaller than the overall scale of the problem; this same disparity of scale is required to invoke the ergodic hypothesis which is implicit in the stochastic approach. Typical vertical and horizontal correlation scales are on the order of 1 and 10 m so that the corresponding overall scale of the problem would be a minimum of 10 and 100 m. Steady flow is assumed as a mathematical convenience and also to focus on the effects of nonlinearity in the Darcy equation of unsaturated flow. For the relatively large vertical scales which are implicit, steady flow is probably a reasonable approximation. A specific goal of the analysis is to determine the mean or effective $K - \psi$ relationship which will apply to a large-scale model of naturally heterogeneous materials.

Part 1 of this series of papers emphasizes the general formulation of the problem and the solution techniques. Specific results are developed for the case of vertical mean infiltration when α is a deterministic constant, with a three-dimensional statistically isotropic and a one-dimensional representation of the hydraulic conductivity variation. The mean behavior of the flow is determined in terms of an effective hydraulic conductivity relationship expressed as a function of mean capillary pressure. Conditions under which the flow becomes one-dimensional are established from comparisons of the variance of capillary pressure head.

In part 2 more general cases with variable α and statistical anisotropy are developed. These analyses explore the anisotropy of effective hydraulic conductivity for unsaturated flow and the effect of the variability of α on the variance of capillary pressure head. In part 3 the new theoretical results are compared with pertinent laboratory and field observations which demonstrate similar behavior. Some practical implications of the results are developed through examples of waste disposal applications.

GENERAL STOCHASTIC FLOW EQUATIONS

In the following we derive the spectral relationship governing steady infiltration in general three-dimensional porous media where both the saturated hydraulic conductivity and the α parameter are considered to be second-order stationary stochastic processes. This general spectral equation is then specialized to consider the one-dimensional and three-dimensional isotropic cases with constant α. The more general cases are treated in part 2.

The general three-dimensional steady flow equation can be written as

$$\frac{\partial}{\partial x_i}\left(K \frac{\partial \Phi}{\partial x_i}\right) = 0 \qquad i = 1, 2, 3 \qquad (2)$$

where K is the hydraulic conductivity (assumed locally isotropic), and $\Phi = -x_i - \psi$ (x_i directed vertically downward). The Einstein summation convention is used. Expanding and dividing by the nonzero conductivity,

$$\frac{\partial^2 \psi}{\partial x_i^2} + \frac{\partial \ln K}{\partial x_1} + \frac{\partial \ln K}{\partial x_i} \frac{\partial \psi}{\partial x_i} = 0 \qquad (3)$$

If the variables are expressed in terms of means and perturbations.

$$\psi = H + h \qquad E[\psi] = H \qquad E[h] = 0 \qquad (4a)$$

$$\alpha = A + a \qquad E[\alpha] = A \qquad E[a] = 0 \qquad (4b)$$

$$\ln K_s = F + f \qquad E[\ln K_s] = F \qquad E[f] = 0 \qquad (4c)$$

Then after neglecting the product of a and h, (1) can be written as

$$\ln K = F + f - AH - Ah - aH \qquad \ln = \log_e \qquad (5a)$$

and

$$\ln K_m = E[\ln K] = F - AH \qquad (5b)$$

Substituting (4) and (5) into (3), the mean flow equation is

$$\frac{\partial^2 H}{\partial x_i^2} + \frac{\partial(F - AH)}{\partial x_1} - \frac{\partial(F - AH)}{\partial x_i} \frac{\partial H}{\partial x_i} + \frac{\overline{\partial ah}}{\partial x_1}$$
$$+ \frac{\overline{\partial(f - Ah - aH - ah)\partial h}}{\partial x_i \, \partial x_i} \approx 0 \qquad (6)$$

Substituting (6) from (3) with F and A assumed constant, yields, after neglecting products of perturbation quantities,

$$\frac{\partial^2 h}{\partial x_i^2} - A(2J_i - \delta_{1i}) \frac{\partial h}{\partial x_i} - J_i(J_i - \delta_{1i})a$$
$$- J_i\left(\frac{\partial f}{\partial x_i} - H \frac{\partial a}{\partial x_i}\right) \approx 0 \qquad (7)$$

Fourier-Stieljes integral representations [Lumley and Panofsky, 1964] are used for the random processes, that is,

$$h(x_1, x_2, x_3) = \int_{-\infty}^{\infty} e^{i k \cdot x} \, dZ_h(k)$$

$$a(x_1, x_2, x_3) = \int_{-\infty}^{\infty} e^{i k \cdot x} \, dZ_a(k) \qquad (8)$$

and

$$f(x_1, x_2, x_3) = \int_{-\infty}^{\infty} e^{i k \cdot x} \, dZ_f(k)$$

where $J_i = -\partial \Phi/\partial x_i$ is the hydraulic gradient, $x = (x_1, x_2, x_3)$ is the position vector, and $k = (k_1, k_2, k_3)$ is the wave number vector. After substitutuion and manipulation, the expression relating the complex Fourier amplitudes of h, a, and f fluctuations is

$$dZ_h = [iJ_n k_n(dZ_f - H \, dZ_a) - (J_n J_n - J_1) \, dZ_a]$$
$$\cdot ([k^2 + iA(2J_n k_n - k_1)])^{-1} \qquad (9)$$

Finally, multiplying both sides of (9) by the complex conjugate of the Fourier amplitude dZ_h, taking mean values, and using the spectral representation theorem produces the spectral relationship

$$S_{hh} = [(J_n k_n)^2(S_{ff} - 2HCo_{af} + H^2 S_{aa}) - 2J_n k_n(J_n^2 - J_1)Q_{af}$$
$$+ (J_n J_n - J_1)^2 S_{aa}][k^4 + A^2(2J_n k_n - k_1)^2]^{-1} \qquad (10)$$

where Q_{af} and Co_{af} are the quadrature and cospectral components of the cross-spectral density function, S_{af}.

Equation (10) is the spectral solution to the stochastic partial differential equations governing the steady state three-

dimensional flow in unsaturated porous media. Particular solutions, which depend on the type of covariance function used to describe the heterogeneity of the medium, will be evaluated in the later sections of this paper and in part 2.

GENERALIZED EFFECTIVE HYDRAULIC CONDUCTIVITY RELATIONSHIP

The general form of effective hydraulic conductivity can be derived from the following specific discharge equation. Assuming local isotropy of the hydraulic conductivity, the Darcy equation for a three-dimensional flow becomes

$$q_i = -K \frac{\partial \Phi}{\partial x_i} = K_m \left[1 + (f - aH - Ah) \right.$$
$$\left. + \frac{(f - aH - Ah)^2}{2} + \cdots \right] \left[J_i + \frac{\partial h}{\partial x_i} \right] \quad (11)$$

where $i = 1, 2,$ and 3. Taking expected values of (11), dropping terms that are beyond the second order, and noting that $E[h\partial h/\partial x_i] = (1/2)\partial E(h^2)/\partial x_i = 0$,

$$E[q_i] = K_m \left[\left(1 + \frac{E[(f - aH - Ah)^2]}{2} \right) J_i \right.$$
$$\left. + E \left((f - aH - Ah) \frac{\partial h}{\partial x_i} \right) \right]$$
$$= K_m \left[\left(1 + \frac{E[(f - aH - Ah)^2]}{2} \right) \delta_{ij} + F_{ij} \right] J_j$$
$$= \bar{K}_{ij} J_j \quad (12)$$

where $E[(f - aH - Ah)\partial h/\partial x_i] = F_{ij}J_j$, $E(f - Ah - aH)^2 = \sigma_f^2 + A^2\sigma_h^2 + H^2\sigma_a^2 - 2AE[fh] - 2HE[af] + 2AHE[ah]$, and δ_{ij} is the Kronecker delta. Thus even though local isotropy is assumed in the derivation of (12), the resulting equation produces a tensorial form of the mean Darcy equation, where $\bar{K}_{ij}$ is the effective hydraulic conductivity tensor. Equation (12) will be used throughout the subsequent parts of the papers to determine the effective unsaturated hydraulic conductivity in one and three-dimensional cases.

STATISTICALLY ISOTROPIC MEDIA WITH CONSTANT α

To illustrate the effect of random $\ln K_s$ on the unsaturated flow, we will assume that the heterogeneous soil can be represented by a statistically isotropic random $\ln K_s$ field and a deterministic (constant) α. Then, S_{aa}, Co_{aa}, and Q_{af} vanish in (10) and only the covariance of the f process is needed to evaluate the statistical properties of the flow system. The mean flow is taken to be vertical infiltration with $J_1 = J$ as the only nonzero component of the mean hydraulic gradient.

One-Dimensional Flow

One-dimensional vertical flow is obtained as a special case of (9) by taking $J_2 = J_3 = 0$, $J_1 = J$, $k_i = k$ and, when $\alpha = $ constant, $dZ_a = 0$,

$$dZ_h = \frac{iJk \, dZ_f}{(k^2 + i\alpha(2J - 1)k)} \quad (13)$$

To evaluate the head variance and covariance, we first use an exponential function for the $\ln K_s$ covariance function, i.e.,

$$R_{ff}(\xi) = \sigma_f^2 \exp[-|\xi|/\lambda] \quad (14)$$

where σ_f^2 is the variance of f, ξ is the separation distance or lag, and λ is the correlation length. The inverse Fourier transform of R_{ff} yields the spectrum of $\ln K_s$,

$$S_{ff}(k) = \frac{\sigma_f^2 \lambda}{\pi(1 + \lambda^2 k^2)} \quad (15)$$

The spectrum S_{hh} of fluctuations in capillary pressure head is derived by using (13) and (15) and the Fourier transform of S_{hh} yields R_{hh}, the autocovariance of head fluctuations:

$$R_{hh}(\xi) = \int_{-\infty}^{\infty} e^{ik\xi} S_{hh}(k) \, dk$$
$$= \frac{J^2 \sigma_f^2}{\lambda} \left[\frac{\lambda^2}{(1 - \lambda^2\beta^2)} \left[\frac{e^{-\beta|\xi|}}{\beta} - \lambda e^{-|\xi|/\lambda} \right] \right] \quad (16)$$

where $\beta = \alpha(2J - 1)$. The variance of capillary pressure σ_h^2 is obtained when ξ is equal to zero:

$$\sigma_h^2 = R_{hh}(0) = \frac{J^2 \sigma_f^2 \lambda}{\beta(1 + \lambda\beta)} \quad (17)$$

Note the head variance approaches infinity as $\lambda\beta$ becomes small.

An alternate form of the autocovariance function of $\ln K_s$ used to produce a statistically homogeneous solution with finite head variance in the stochastic analysis of one-dimensional, saturated groundwater flow [Bakr et al., 1978], is a "hole" covariance function

$$R_{ff}(\xi) = \sigma_f^2 \left[1 - \frac{|\xi|}{\eta} \right] \exp[-|\xi|/\eta] \quad (18)$$

where η is the correlation length. The spectrum of $\ln K_s$, associated with this autocovariance function is

$$S_{ff}(k) = \frac{2\sigma_f^2 \eta^3 k^2}{\pi(1 + \eta^2 k^2)^2} \quad (19)$$

and the autocovariance of head fluctuations resulting from (13) is

$$R_{hh}(\xi) = \frac{J^2 \sigma_f^2 \eta^2}{(1 - \beta^2\eta^2)^2} \left[[1 - \beta^2\eta^2] \left[1 + \frac{|\xi|}{\eta} \right] e^{-|\xi|/\eta} \right.$$
$$\left. - 2\beta\eta e^{-\beta|\xi|} + 2\beta^2\eta^2 e^{-|\xi|/\eta} \right] \quad (20)$$

where $\beta = \alpha(2J - 1)$. This head covariance function and the hydraulic conductivity covariance function are graphed in Figure 1. Note that the autocorrelation function of the soil capillary pressure head depends not only on the ratio of the separation distance to the correlation scale of $\ln K_s$, as in saturated flow case [Bakr et al., 1978], but also on the gradient, J, and $\alpha\eta$. The corresponding head variance is

$$\sigma_h^2 = R_{hh}(0) = \frac{J^2 \sigma_f^2 \eta^2}{(1 - \beta\eta)^2} \quad (21)$$

The head variance resulting from the exponential and hole covariance functions are shown in Figure 2 as a function of $\alpha\lambda$ using $\eta = 2.5\lambda$ [Bakr et al., 1978].

Three-Dimensional Flow

From (10) with $J_n k_n = Jk_1$ and $\alpha = $ const, the spectral relationship for a three-dimensional flow situation is

$$S_{hh} = \frac{J^2 k_1^2 S_{ff}}{(k^2 + \alpha^2(2J - 1)^2 k_1^2)} \quad (22)$$

Considering a simple exponential form of the autocovariance function for f [see Bakr et al., 1978], namely,

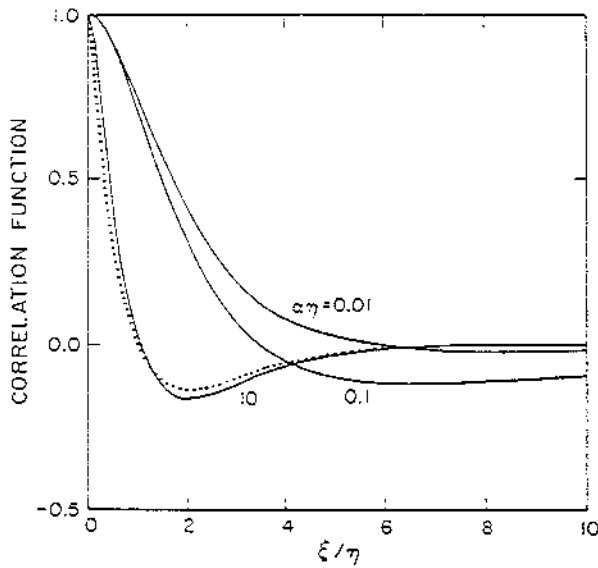$$R_{ff}(\xi) = \sigma_f^2 \exp[-\xi/\lambda] \quad (23)$$

Fig. 1. Correlation functions of head (20) for different $\alpha\eta$ values (solid curve) and ln K_s (dotted curve) (18) versus the dimensionless lag ξ/η; $J = 1$.

where σ_f^2 is the variance of ln K_s, $\xi = [\xi_1^2 + \xi_2^2 + \xi_3^2]^{1/2}$ is the length of the separation vector, and λ is the integral scale. The corresponding spectrum of ln K_s [Bakr et al., 1978] is

$$S_{ff} = \frac{\sigma_f^2 \lambda^3}{\pi^2 (1 + k^2 \lambda^2)^2} \tag{24}$$

and hence the spectrum of head fluctuations is

$$S_{hh} = \frac{J^2 \sigma_f^2 \lambda^3 k_1^2}{\pi^2 (1 + k^2 \lambda^2)^2 (k^4 - \beta^2 k_1^2)} \tag{25}$$

where $\beta = \alpha(2J - 1)$.

The head variance is found by integrating the spectrum (25) over wave number space (see the appendix) and is given by

$$\sigma_h^2 = \frac{J^2 \sigma_f^2 \lambda^2}{(\beta\lambda)^2} \left[1 - \frac{2 \ln (1 + \lambda\beta)}{\lambda\beta} + \frac{1}{1 + \lambda\beta} \right] \tag{26}$$

The head covariance function for the three-dimensional case is more complicated, but it can be obtained by taking the Fourier transform of S_{hh},

$$R_{hh}(\xi, \chi) = \iiint\limits_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\xi} S_{hh}(\mathbf{k}) \, d\mathbf{k} \tag{27}$$

where χ is the angle between the separation vector ξ and the direction of mean flow, and $\xi = |\xi|$. Transforming $\mathbf{k}$ into spherical coordinates following Bakr et al. [1978],

$$q = \frac{k_1}{k} = \cos \Phi \cos \chi + \sin \Phi \sin \chi \cos \theta$$

$$d\mathbf{k} = k^2 \sin \Phi \, dk \, d\Phi \, d\theta \tag{28}$$

and integrating over k (27) becomes

$$R_{hh}(\xi, \chi) = C\pi \int\limits_{\theta=0}^{2\pi} \int\limits_{\Phi=0}^{\pi} \frac{q^2 \sin \Phi}{(b^2 q^2 - 1)} \left[\frac{-bq e^{-bq|\tau|}}{(b^2 q^2 - 1)} \right.$$
$$\left. + \frac{b^2 q^2 e^{-|\tau|}}{(b^2 q^2 - 1)} - \frac{(|\tau| + 1) e^{-|\tau|}}{2} \right] d\theta \, d\Phi \tag{29}$$

where $q = k_1/k$, $b = \beta\lambda$, $\tau = (\xi/\lambda) \cos \varphi$, and $C = J^2 \sigma_f^2 \lambda^2$. The remaining integral is evaluated numerically. The result for the head covariance function is graphed in Figure 3.

Variance of Log Unsaturated Hydraulic Conductivity

If α is assumed to be a deterministic constant, (5a) can be simplified to ln $K = F + f - \alpha(H + h)$, and the corresponding equation describing the fluctuation of ln K is ln $K - E[\ln K] = f - \alpha h$. The variance of ln K then can be directly obtained by

$$\sigma_{\ln K}^2 = E[f^2] + \alpha^2 E[h^2] - 2\alpha E[fh] \tag{30}$$

In one dimension,

$$E[fh] = \int_{-\infty}^{\infty} E[dZ_f \, dZ_h^*] \, dk$$
$$= \int_{-\infty}^{\infty} \frac{J\beta S_{ff}}{k^2 + \beta^2} \, dk = \frac{\beta}{J} \sigma_h^2$$

because (13) yields

$$\sigma_h^2 = \int_{-\infty}^{\infty} \frac{J^2 S_{ff}}{k^2 + \beta^2} \, dk$$

Similarly, in three dimensions,

$$E[fh] = \int_{-\infty}^{\infty} \frac{ik_1 J S_{ff}}{k^2 - i\beta k_1} \, dk$$
$$= \int_{-\infty}^{\infty} \frac{J\beta k_1^2}{k^2 + \beta^2 k_1^2} S_{ff} \, dk = \frac{\beta}{J} \sigma_h^2$$

where (22) is used to get σ_h^2. Thus independent of the dimension and the form of the input spectrum, we get

$$\sigma_{\ln K}^2 = \sigma_f^2 + [\alpha^2 - 2(\alpha\beta/J)]\sigma_h^2 \tag{31}$$

Substituting (16) and (17) for the one-dimensional head variance in (31), the variances of one-dimensional unsaturated flow hydraulic conductivity for a simple exponential auto-covariance function (14) and a hole function (18) are
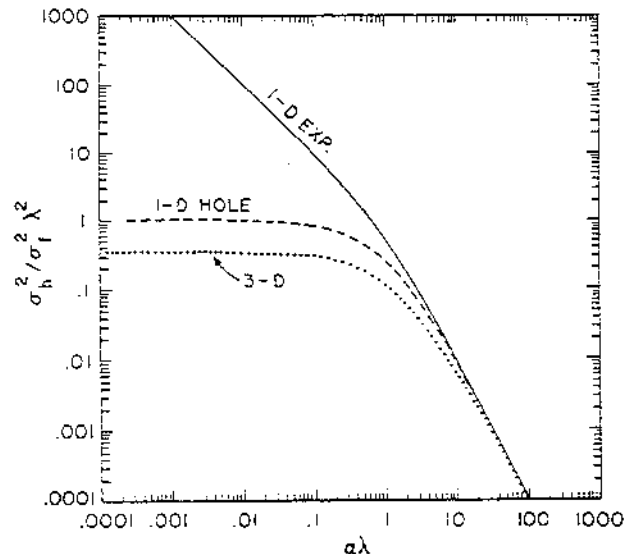

Fig. 2. Comparison of the one-dimensional head variance results for the exponential and hole function, (17) and (21), respectively, with $\eta = 2.5\lambda$ and the three-dimensional head variance (26) for an isotropic medium; $J = 1$.

$$\sigma_{\ln K}^2 = \sigma_f^2\left[1 - \left(x^2 - 2\frac{x\beta}{J}\right)\frac{J^2\lambda}{\beta(1 + \lambda\beta)}\right] \quad (32a)$$

$$\sigma_{\ln K}^2 = \sigma_f^2\left[1 - \left(x^2 - 2\frac{x\beta}{J}\right)\frac{J^2\eta^2}{(1 + \beta\eta)^2}\right] \quad (32b)$$

respectively.

The variance of unsaturated hydraulic conductivity for the three-dimensional model, using (26) and (31) is

$$\sigma_{\ln K}^2 = \sigma_f^2\left[1 - \left(x^2 - \frac{2x\beta}{J}\right)\frac{J^2\lambda^2}{(\beta\lambda)^2}\right.$$
$$\left. \cdot\left[1 - \frac{2\ln(1 + \lambda\beta)}{\lambda\beta} + \frac{1}{1 + \lambda\beta}\right]\right] \quad (33)$$

Effective Unsaturated Hydraulic Conductivity

When z is a deterministic constant, from (12), the mean specific discharge of a one-dimensional flow case $(x_1 = z, q_1 = q, J_1 = J, J_2 = J_3 = 0, \partial h/\partial x_1 = \partial h/\partial x_3 = 0)$ becomes

$$E[q] \approx K_m\left[1 + \frac{1}{2}E[(f - xh)^2]\right]J + K_mE\left[(f - xh)\frac{dh}{dz}\right] \quad (34)$$

where $K_m = K_G \exp(-xH)$, $\ln K_G = E[\ln K_s]$, and $J = 1 - dH/dZ$ is the mean hydraulic gradient. The first expected value term on the right hand side of (34) is simply the variance of the logarithm of the unsaturated hydraulic conductivity, $\sigma_{\ln K}^2$. The second expected value in (34) can be determined by noting $E[hdh/dz] = 0$ for any stationary process and by using

$$E\left[f\frac{dh}{dz}\right] = \int_{-\infty}^{\infty} - ikE(dZ_h^* \, dZ_f) \, dk$$
$$= \int_{-\infty}^{\infty}\frac{-Jk^2}{k^2 + \beta^2}S_{ff} \, dk$$
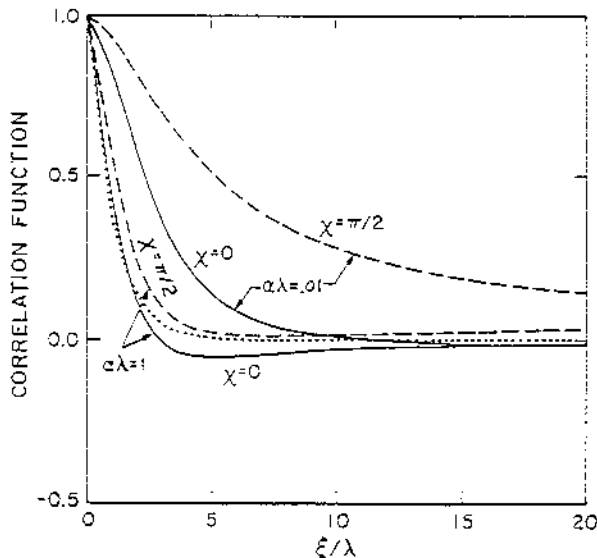$$= -J\int_{-\infty}^{\infty}\left[1 - \frac{\beta^2}{k^2 + \beta^2}\right]S_{ff} \, dk$$

Fig. 3. Correlation function (29) of three-dimensional head perturbations at different angles from the mean flow (solid and dashed curves) compared with that of $\ln K_s$ (dotted curves); $J = 1$.

$$= -J\left[\sigma_f^2 - \frac{\beta^2}{J^2}\sigma_h^2\right]$$

for any input spectrum S_{ff}. Thus

$$E[q]/J = K_m\left[1 + \frac{\sigma_f^2}{2}\left[1 + \left(x^2 - \frac{2x\beta}{J}\right)\frac{J^2\lambda}{\beta(1 + \lambda\beta)}\right.\right.$$
$$\left.\left. - \frac{2}{1 + \lambda\beta}\right]\right] \equiv K_e \quad (35)$$

$$E[q]/J = K_m\left[1 + \frac{\sigma_f^2}{2}\left[1 + \left(x^2 - \frac{2x\beta}{J}\right)\frac{J^2\eta^2}{(1 + \beta\eta)^2}\right.\right.$$
$$\left.\left. - \frac{2(1 + 2\beta\eta)}{(1 + \beta\eta)^2}\right]\right] \equiv K_e$$

for the exponential and the hole function covariances, respectively.

The right-hand side of (35), K_e, multiplied by J, the mean hydraulic gradient, produces the mean moisture flux or specific discharge. Note K_e generally depends on the mean hydraulic gradient, i.e., the mean Darcy equation is nonlinear. When $J = 1$, $(\beta = a(2J - 1) = x)$, (35) implies a relationship between the mean infiltration rate and the mean capillary pressure. Since $K_m = K_G \exp(-xH)$ and $K_G = \exp E[\ln K_s]$, the geometric mean of the saturated hydraulic conductivity, the effective conductivity of the unsaturated flow reduces to

$$K_e = K_G e^{-xH}\left[1 - \frac{\sigma_f^2}{2(1 - x\lambda)}\right] \quad (36a)$$

Using the relationships in (32a) with $J = 1$, this can be expressed in terms of $\sigma_{\ln K}^2$ as

$$K_e = K_G e^{-xH}\left[1 - \frac{\sigma_{\ln K}^2}{2}\right] \quad (36b)$$

Equation (36b) represents the mean or effective conductivity-capillary pressure relationship for steady vertical infiltration through a perfectly stratified heterogeneous soil of unbounded vertical extent.

Note that if $\sigma_f^2/[2(1 + x\lambda)] > 1$, K_e becomes negative. This unreasonable result occurs because higher order terms are neglected in (12) and (7). For the saturated flow case, *Gutjahr et al.* [1978] found an 18% error in this type of first order approximation for one-dimensional flow perpendicular to layering with a lognormally distributed K and $\sigma_f^2 = 1$. For the unsaturated flow case, the error due to the approximation depends on σ_f^2, x, and λ for $J = 1$. The magnitude of $\sigma_f^2/[2(1 + x\lambda)]$ is likely to be greater than 1 for some soils. One possible way to extrapolate to these relationships to large σ_f is to consider the quantities in brackets in (35) and (36) as the first two terms of the Taylor series expansion for e^x as proposed by *Gelhar and Axness* [1983]; then K_e becomes

$$K_e = K_G \exp\left\{-xH - [\sigma_f^2/2(1 + x\lambda)]\right\} \quad (37)$$

The more general formula (35) in which J is not restricted to be 1, then can be expressed as

$$K_e = K_G \exp\left[-xH + \frac{\sigma_f^2}{2}\left[1 + \left(x^2 - \frac{2x\beta}{J}\right)\frac{J^2\lambda}{\beta(1 + \lambda\beta)}\right.\right.$$
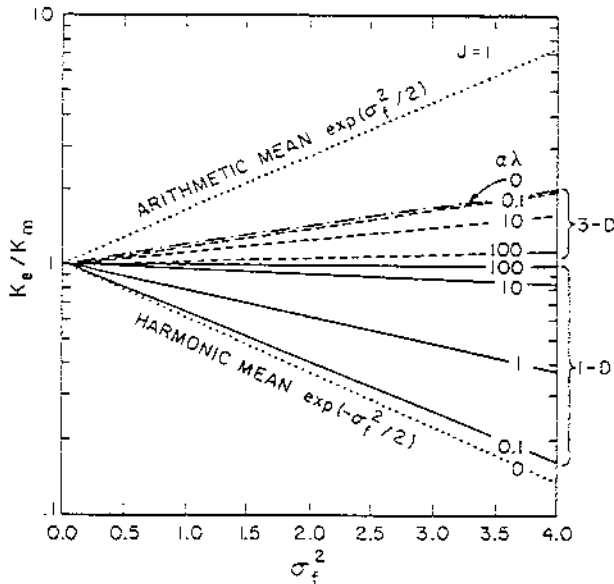$$\left.\left. - \frac{2}{1 + \lambda\beta}\right]\right] \quad (38)$$

Fig. 4. Ratio of the effective hydraulic conductivities (39) and (43) to the geometric mean of unsaturated hydraulic conductivity, $K_m = K_G \exp(-\alpha H)$; $\alpha\lambda = 0$ corresponds to saturated flow.

The same exponential generalization for the hole covariance function yields

$$K_e = K_G \exp\left[-\alpha H + \frac{\sigma_f^2}{2}\left[1 + \left(\alpha^2 - \frac{2\alpha\beta}{J}\right)\frac{J^2\eta^2}{(1 + \beta\eta)^2}\right.\right.$$
$$\left.\left. - \frac{2(1 + 2\beta\eta)}{(1 + \beta\eta)^2}\right]\right] \qquad (39)$$

A similar analysis is carried out for the three-dimensional case. The procedure is identical to that of the one-dimensional case although the integrations are more involved. If the mean gradient exists only in the x_1 direction, the mean Darcy flux (12) in this direction takes the form

$$E[q_1] = K_m\left[1 + \frac{E[(f - \alpha h)^2]}{2}\right]J_1 - E\left[(f - \alpha h)\frac{\partial h}{\partial x_1}\right] \qquad (40)$$

where $E[(f - \alpha h)^2] = \sigma_{\ln K}^2$. Since the covariance of h and $\partial h/\partial h_1$ is zero, the last term of the right hand side of (40) can be determined as follows:

$$E[fj] = E\left[f\frac{\partial h}{\partial x_1}\right] = \iiint_{-\infty}^{\infty} S_{fj}(\mathbf{k})\, d\mathbf{k}$$

$$= \frac{-J_1\sigma_f^2\lambda^3}{\pi^2}\iiint_{-\infty}^{\infty}\frac{(k_1^2 k^2 + ik_1^3\beta)\, dk_1\, dk_2\, dk_3}{(k^4 + k_1^2\beta^2)(k^2\lambda^2 + 1)^2}$$

$$E[fj] = J_1\sigma_f^2\left[\frac{2}{y^2}\left[1 - \frac{2\ln|1 - y|}{y} + \frac{1}{1 + y}\right] - \frac{1}{1 - y}\right] \qquad (41)$$

where $y = \lambda\beta$ (details of this integration are given in the work by Yeh [1982]). Thus the effective hydraulic conductivity for the three-dimensional flow case is

$$E[q_1]/J_1 = K_e = K_G e^{-\alpha H}\left[1 + \frac{\sigma_{\ln K}^2}{2} - \frac{E[fj]}{J_1}\right] \qquad (42)$$

Assuming $J_1 = 1$ and using the exponential approach given by (37) and (42) results in

$$K_e = K_G \exp\left[-\alpha H + \sigma_f^2\left[\frac{\ln(1 + \lambda\alpha)}{\alpha\lambda} - \frac{1}{2(1 - \alpha\lambda)} + \frac{3}{(\alpha\lambda)^2}\right.\right.$$
$$\left.\left. - \frac{1}{\alpha\lambda} - \frac{4\ln(1 + \alpha\lambda)}{(\lambda\alpha)^3} + \frac{1}{(\alpha\lambda)^2(1 + \alpha\lambda)}\right]\right] \qquad (43)$$

This equation describes the effective hydraulic conductivity in a three-dimensional steady state infiltration in statistically isotropic unsaturated porous media.

Figure 4 shows the effective hydraulic conductivities (39) and (43) of the one- and three-dimensional models as a function of the magnitude of the σ_f^2. Note that the effective hydraulic conductivity also depends on the mean gradient.

Flux Variance

Subtracting (40) from (11) and neglecting the higher-order terms, the equation describing the perturbation of the flux is of the form

$$q_1' \approx K_m\left[J(f - \alpha h) + \frac{\partial h}{\partial x_1}\right] \qquad (44)$$

If the perturbed term in (44) is represented by the stochastic Fourier-Stieltjes integral, the relationship among the Fourier amplitudes of the perturbations q_1', f, and h is

$$dZ_{q_1} = K_m[J\, dZ_f - (\alpha J - ik_1)\, dZ_h] \qquad (45)$$

and the corresponding spectral relationship is

$$S_{q_1} = K_m^2[J^2 S_{ff} - 2\alpha J^2\, \text{Re}\,[S_{fh}] + 2J\, \text{Re}\,[S_{fj}]$$
$$+ (\alpha^2 J^2 + k_1^2)S_{hh}] \qquad (46)$$

where Re denotes the real part of the spectrum. The variance of q_1' can be obtained by integrating (46) over wave number with the specified covariance function for the f process, (23). The resulting variance of q_1' is

$$\sigma_{q_1}^2 = K_m^2\sigma_f^2 J^2\left(1 - \frac{3}{2\lambda\beta} + \frac{5}{(\lambda\beta)^2} + \frac{2}{(\lambda\beta)^3} - \frac{5}{(\lambda\beta)^4}\right.$$
$$+ \frac{\ln(1 + \lambda\beta)}{(\lambda\beta)^3}\left[-8 + \frac{6}{(\lambda\beta)^2}\right] + \frac{2}{(\lambda\beta)^2(1 + \lambda\beta)}$$
$$- \frac{1}{(\lambda\beta)^4(1 + \lambda\beta)} - \frac{\alpha^2\lambda^2 J(3J - 2)}{(\lambda\beta)^2}$$
$$\left. \cdot\left[1 - \frac{2\ln(1 + \lambda\beta)}{\lambda\beta} - \frac{1}{1 + \lambda\beta}\right]\right) \qquad (47)$$
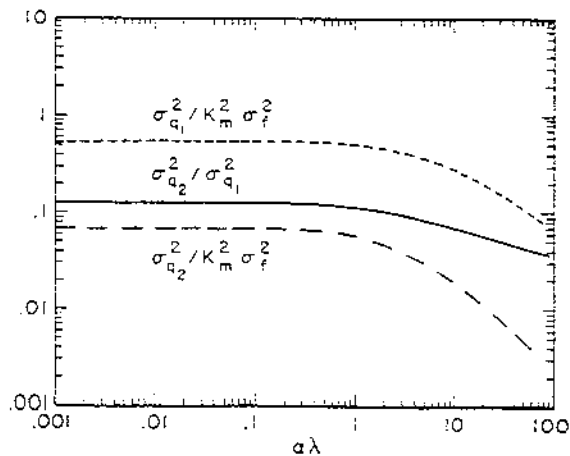

Fig. 5. Comparison of normalized longitudinal and transverse flux variances of the three-dimensional flows; $J = 1$.
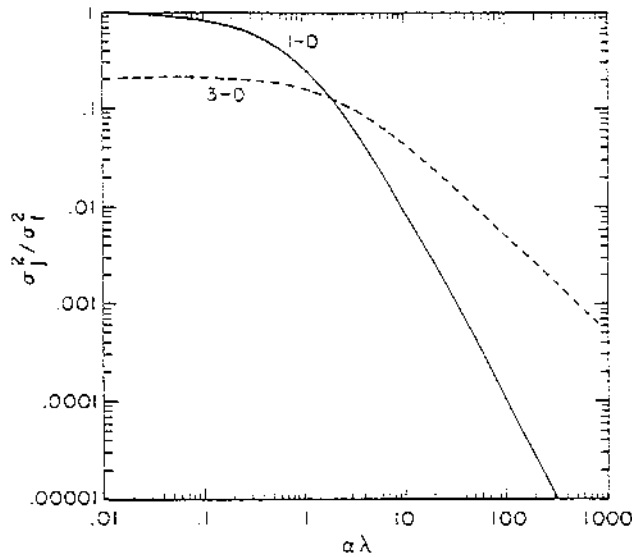
Fig. 6. Comparison of one- and three-dimensional gradient variances, (54) and (55), respectively; $\eta = 2.5\lambda$, $J = 1$.

The variance of flux perpendicular to the mean gradient J_t is evaluated as follows:

$$q_2' = K_m e^{(f - xh)} \frac{\partial h}{\partial x_2}$$

$$\approx K_m (1 + (f - xh) + \cdots) \frac{\partial h}{\partial x_2} \cong K_m \frac{\partial h}{\partial x_2} \qquad (48)$$

to first order in perturbations. The corresponding spectral relationship of perturbations of q_2 and h is of the form

$$S_{q_2 q_2} = K_m^2 (k_2^2 S_{hh}) \qquad (49)$$

The variance of q_2 is obtained by integrating (49) over wave number space, with the result

$$\sigma_{q_2}^2 = \frac{K_m^2 J^2 \sigma_f^2}{2} \left[\frac{6}{(\lambda\beta)^4} - \frac{3}{(\lambda\beta)^3} - \frac{2}{(\lambda\beta)^2} + \frac{1}{2\lambda\beta} \right.$$

$$\left. + \frac{\ln(1 + \lambda\beta)}{(\lambda\beta)^3} \left[4 - \frac{6}{(\lambda\beta)^2} \right] \right] \qquad (50)$$

To demonstrate the effects of $x\lambda$ on the results, $\sigma_{q_1}^2$, and $\sigma_{q_2}^2$ are evaluated as a function of $x\lambda$ for the case of $J = 1$ and are displayed in Figure 5. Note that for small $x\lambda$ the ratio $\sigma_{q_2}^2 / \sigma_{q_1}^2$ is $\frac{1}{3}$, the saturated flow value. For large $x\lambda$, reflecting unsaturated flow effects, this ratio decreases; this trend of decreasing transverse flow variation shows the tendency toward one-dimensional flow.

Variance of Pressure Gradient

In analyses of infiltration and drainage, it is often assumed that capillary pressure gradient is negligible (unit hydraulic gradient). One may question the validity of this assumption, especially in soils exhibiting a large degree of spatial variation. The evaluation of pressure gradient variation through this theoretical analysis provides a quantitative estimate of the variation of the pressure gradient in a heterogeneous soil.

For one-dimensional flow the Fourier amplitude relationships between pressure gradient $j = dh/dz$ and pressure head is given as

$$dZ_j = ik \, dZ_h \qquad (51)$$

Multiplying (51) by its complex conjugate and then taking its expected value results in the spectral relationship.

$$S_{jj} = k^2 S_{hh} = \frac{k^2 S_{ff}}{k^2 + \beta^2} \qquad (52)$$

To evaluate the variance of the j process, its spectrum is integrated. For exponential covariance function, using (15),

$$\sigma_j^2 = \frac{J^2 \sigma_f^2}{\lambda\beta(1 + \lambda\beta)} \qquad (53)$$

and for the hole function, (19),

$$\sigma_j^2 = \frac{J^2 \sigma_f^2}{(1 + \beta\eta)^2} \qquad (54)$$

Similarly, the variance of the pressure gradient resulting from the three-dimensional model can be evaluated [see Yeh, 1982] as

$$\sigma_j^2 = J^2 \sigma_f^2 \left[\frac{1}{2y} - \frac{1}{y^2} + \frac{2}{y^3} - \frac{5}{y^4} - \frac{1}{y^5} \left(\frac{y}{1 + y} \right. \right.$$

$$\left. \left. - 6 \ln(1 + y) \right) \right] \qquad (55)$$

where $y = \lambda\beta$. These results are illustrated graphically in Figure 6.

DISCUSSION OF THE RESULTS

Figure 2 shows the normalized head variance at $J = 1$ as a function of the product $x\lambda$. To compare the effect of the ln K_s covariance functions, the length scale η of the hole function in the one-dimensional case is taken to be $\eta = 2.5\lambda$, where λ is the correlation scale of the simple exponential [Bakr et al., 1978]. Physically, x^{-1} can be considered as the thickness of the capillary fringe. Typical ranges for x^{-1} and λ [see Bouwer, 1964, 1978; Bakr, 1976] are 0.2–2 m for both parameters; therefore the practical range of $x\lambda$ would be from 0.1 to 10. Figure 2 shows that, at the upper end of this range, the two one-dimensional input covariance functions, the exponential and the hole function, produce practically the same head variance. For a small value of $x\lambda$ the head variance resulting from the exponential function approaches infinity as indicated by the one to one slope of the curve, whereas the hole function produces a finite head variance. In addition, the head variances obtained from the two input functions at the limit $(x \rightarrow 0)$ are consistent with that of saturated flow [see Bakr et al., 1978]. For $x = 0$ $(A = a = 0)$ the flow equation (7) (or equation (13) for one-dimensional flow) is equivalent to the saturated flow case of Bakr et al. [1978].

The effect of large $x\lambda$ can be interpreted in terms of the governing one-dimensional steady state infiltration equation (from equation (7) for this special case) expressed in terms of capillary pressure head and log saturated conductivity perturbations

$$\frac{d^2h}{dz^2} + \frac{dH}{dz}\frac{df}{dz} - 2x \frac{dH}{dz}\frac{dh}{dz} + \left[\frac{df}{dz} - x \frac{dh}{dz} \right] = 0 \qquad (56)$$

This equation can be compared to the horizontal absorption case

$$\frac{d^2h}{dz^2} + \frac{dH}{dz}\frac{df}{dz} - 2x \frac{dH}{dz}\frac{dh}{dz} = 0 \qquad (57)$$

The additional term on the left-hand side of (56) represents the

gravity term consisting of the effects of the hydraulic conductivity and the capillary pressure head perturbation gradients. The relationship between head variance and α becomes evident. When α becomes large, the effect due to the variation in saturated conductivity is significantly reduced by the head perturbation gradient, which is amplified by the magnitude α. In other words, the gravity term becomes dominant and the head variance is reduced.

The reduction of head variance at large value of $\alpha\lambda$ can be further elaborated through a "conditional analysis." Assume that soil formation is composed of a collection of soil columns for which the interaction between columns is neglected. Each individual column is assumed to be homogeneous and Darcy equation for each column is

$$q = K(\psi)\left(\frac{d\psi}{dz} + 1\right)$$

For an unbounded column $d\psi/dz = 0$ and

$$q = K(\psi) = K_s e^{-\alpha\psi}$$

This equation yields the expression for the soil capillary pressure head

$$\psi = \frac{\ln K_s}{\alpha} - \frac{\ln q}{\alpha}$$

and if the flux is taken to be a specified (deterministic) constant, the variance of capillary pressure head is of the form

$$\sigma_\psi^2 = \frac{\sigma_f^2}{\alpha^2}$$

Note that the head variance in the above equation is a result of variations in saturated hydraulic conductivities among soil columns, since each column is homogeneous, and the correlation scale of the saturated hydraulic conductivity of each column is infinite.

This same result is found from the stochastic analysis, (17), when $\beta\lambda = \alpha\lambda \gg 1$ with $J = 1$; however, this case could also be interpreted as one with fixed λ much smaller than the overall scale of the problem and $\alpha\lambda$ large because of the soil type. Both interpretations demonstrate the head variance reduction when α is large. Note that when $\alpha\lambda \gg 1$ the stochastic result, (17), shows the head variance is independent of the correlation scale. This behavior is in contrast to the saturated flow case where the head variance increases as λ^2 [Bakr et al., 1978].

The one-dimensional solutions for both the exponential and the hole covariance are seen to yield finite variance, stationary head solutions when α is nonzero. This is in contrast to the saturated flow situation where the stationarity occurs only for hole covariances [Gutjahr and Gelhar, 1981]. Stationary behavior was also found for one-dimensional unsaturated flow by Andersson and Shapiro [1983]; they also showed good agreement between a perturbation solution in the space domain and Monte Carlo simulations even with relatively large variability of the saturated hydraulic conductivity.

Figure 2 also shows that the head variance derived from the three-dimensional model approaches the saturated flow result as $\alpha\lambda$ approaches zero. For large values of $\alpha\lambda$, the one- and three-dimensional results are identical, indicating that the flow is predominantly one-dimensional under this condition. Therefore the one-dimensional result may be appropriate for some field applications, especially for coarse-textured soils

which generally are associated with large values of α. For fine-textured materials such as clay and silt, which are often characterized by small values of α, the one-dimensional results may not be appropriate. The three-dimensional analysis produces a smaller head variance in these types of soils. Hence for fine-textured soils, significant errors could be introduced if results from a one-dimensional model are used to draw a conclusion about the effects of field heterogeneity.

The head covariance functions resulting from the one-dimensional analysis are shown in Figure 1. This figure demonstrates that the mathematical filtering effect is directly related to the magnitude of $\alpha\lambda$. When $\alpha\lambda$ is small, the head perturbations tend to be correlated over a large distance. Conversely, the output covariance function tends to have the same correlation length scale as that of the input covariance when $\alpha\lambda$ is large.

In Figure 3 the head covariance function of the three-dimensional model is evaluated for several different values of the angle χ and $\alpha\lambda$. The head perturbation in this case is anisotropic, even though the input log saturated hydraulic conductivity perturbation is isotropic. The perturbations of the head process in the direction perpendicular to the mean flow ($\chi = \pi/2$) have consistently higher correlation values than in the direction parallel to the mean flow, particularly for small values of $\alpha\lambda$. However, the difference between the correlation values in the two directions diminishes as $\alpha\lambda$ becomes large. Furthermore, the fluctuations in head in the direction parallel to the mean flow direction tend to exhibit the hole effect at large $\alpha\lambda$ values, when the exponential covariance function is used. This further confirms the finding that the infiltration process in coarse-textured soils is simply a one-dimensional phenomenon. Conversely, in fine-textured soils, water will dissipate laterally in response to the lateral head perturbation gradient and the process becomes three-dimensional.

The variance of log-unsaturated hydraulic conductivity as indicated by (32) and (33) is found to decrease with $\lambda\beta$. This is consistent with the previous finding on head variance in unsaturated flow.

The above interpretations are straightforward for the case of vertical mean infiltration in an unbounded region because the mean hydraulic gradient is constant ($J_1 = 1$) in that case. However, for bounded cases, such as flow approaching a water table, the mean gradient is not constant, and the applicability of the local-stationarity assumption will be restricted to situation where the mean gradient varies slowly over a correlation scale. The scale of variation of the mean solution will be of order α^{-1} and local stationarity can be expected only if $\lambda \ll \alpha^{-1}$. As a consequence, the stationary results are not expected to be valid for $\alpha\lambda \gg 1$ when J is not constant.

The effective hydraulic conductivities determined from the one- and three-dimensional models are shown graphically in Figure 4. It is seen that the effective hydraulic conductivities approach the one- and three-dimensional saturated flow limits found by Gutjahr et al. [1978] as $\alpha\lambda$ becomes small. In addition, they are bounded by the harmonic and arithmetic means of the saturated hydraulic conductivities. Note also that for large $\alpha\lambda$ both one- and three-dimensional effective hydraulic conductivities approach $K_G e^{-\alpha H}$. The effective saturated conductivity is simply the geometric mean under this condition. For small $\alpha\lambda$ the trends in Figure 4 can be thought of simply as the multiplier of geometric mean required to yield the effective saturated conductivity; for the three-dimensional case

the multiplier is greater than one and for one-dimensional flow it is less than one.

As expected, the variance of the flux in the lateral direction diminishes relative to that in the direction of the mean flow as x becomes large (Figure 5). This is also an indication of the dominance of gravitational flow in coarse-textured soil.

The variation of pressure gradient also can be related to $x\lambda$ is illustrated in Figure 6. For small $x\lambda$ the one-dimensional analysis produces a variance of $J^2\sigma_f^2$, and the three-dimensional analysis results in a smaller capillary pressure gradient variation which is $\frac{1}{2}$ of the one-dimensional result. As the value of $x\lambda$ increases, the variance resulting from one-dimensional analysis decreases more rapidly than that of the three dimensional case. These results indicate the assumption of a unit hydraulic gradient may be reasonable for coarse textured soils with a large $x\lambda$ but will be questionable for soil with a small $x\lambda$.

APPENDIX: EVALUATION OF HEAD VARIANCE σ_h^2 IN (33)

This appendix describes the integration procedures used in obtaining the head variance expression given in (26). The same approach was also used to evaluate other integrals necessary to determine other variance and covariance terms in this part of the paper. Details of these integration procedures are given by [Yeh, 1982].

To obtain the head variance, one integrates the spectrum of the head fluctuation given in (25), which is written as follows:

$$\sigma_h^2 = \frac{J^2\sigma_f^2}{\pi^2\lambda} \int\!\!\!\int\!\!\!\int_{-\infty}^{\infty} \frac{k_1^2 \, dk_1 \, dk_2 \, dk_3}{(k^4 + \beta^2 k_1^2)(k^2 + 1/\lambda^2)^2} \quad (A1)$$

Expressing k_i ($i = 1, 2,$ and 3) by the following spherical coordinates, $k_1 = k \cos \Phi$, $k_2 = k \sin \Phi \sin \theta$, $k_3 = k \sin \Phi \cos \theta$, and letting $t = \cos \Phi$ and integrating over Θ, the resulting integrand is

$$\frac{4J^2\sigma_f^2}{\pi\lambda} \int_0^1 \int_0^{\infty} \frac{k^2 t^2 \, dk \, dt}{(k^2 + 1/\lambda^2)^2 (k^2 + \beta^2 t^2)} \quad (A2)$$

By letting $V = \beta t$, and integrating (A2) over k by partial fraction, (A2) can be reduced to

$$\frac{J^2\sigma_f^2}{\beta^3} \int_0^{\beta} \frac{V^2 \, dV}{(1/\lambda + V)^2} \quad (A3)$$

The integration over V will result in the expression for the head variance given in (26).

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Correction to "Stochastic Analysis of Unsaturated Flow in Heterogeneous Soils, 1 and 2," By T.-C. J. Yeh, L. W. Gelhar, and A. L. Gutjahr

In the paper "Stochastic Analysis of Unsaturated Flow in Heterogeneous Soils, 1 and 2" by T. C. Yeh et al. (*Water Resources Research*, *21*(4), 447–456 and 457–464, 1985), an important correction in some equations has been noticed.

The statement right after equation (4c) on p. 448, "Then after neglecting the product of a and h ..." should be deleted. Equation (5a) ought to be

$$\ln K = F + f - Ah - aH - ah \qquad \ln = \log_e \qquad (5a)$$

and (5b) is

$$\ln K_m = E[\ln K] = F - AH - E[ah] \qquad (5b)$$

In addition, equation (11) should be written as

$$q_1 = -K \frac{\partial \phi}{\partial x_i} = \exp(F - AH)\left[1 + (f - aH - Ah - ah) \right.$$

$$\left. + \frac{(f - aH - Ah - ah)^2}{2} + \cdots \right]\left[J_i + \frac{\partial h}{\partial x_i} \right] \qquad (11)$$

Note that the effects of the additional term $E[ah]$ are incorporated in subsequent equations simply by expressing K_m as in the corrected (5b). This additional term is nonzero only for the variable α case (part 2), in which case the required $E[ah]$ is given in Table 1. This correction does not affect the anisotropy ratio calculations in part 3 because it is an isotropic effect.

In equation (6a), on p. 458, ≥ 0 should be > 0.

In equation (6b), on p. 458, ≤ 0 should be < 0. And if $[\rho^4 g^2 - 4(\rho^2 - 1)] = 0$ where $a = \rho^2 g$ and $b = 2(\rho^2 - 1)$.

$$\sigma_h{}^2 = J_1{}^2 \sigma_f{}^2 \lambda_1{}^2 \rho^4 / [3a(a + b)^3 b^4]$$

In Table 1, on p. 461, the $E[aj_m]$ for case 2 should be positive, and, on the first line, $\sigma_n{}^2$ should be replaced by $\sigma_h{}^2$.