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BURGERS' EQUATION AND LAYERED MEDIA: EXACT SOLUTIONS AND APPLICATIONS TO SOIL-WATER FLOW

PHILIP BROADBRIDGE
Department of Mathematics, University of Wollongong
Wollongong 2500, Australia
RAJESH SRIVASTAVA AND T.-C. JIM YEH
Department of Hydrology and Water Resources, University of Arizona
Tucson, Arizona 85721, U.S.A.

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Abstract—Exact solutions are presented for Burgers' equation in a finite layer connected to an underlying semi-infinite medium of different conductivity and diffusivity. A constant-flux boundary condition is assumed at the surface. This has direct application to steady rainfall on layered field soils. At large times, a travelling wave profile develops in the deep layer and the concentration in the upper layer approaches a non-trivial steady state. Water flux and potential energy are continuous across the interface but the concentration gradient may show a marked discontinuity.

1. INTRODUCTION

Burgers' equation

$$v_t = Dv_{xx} + 2Evv_x, \qquad D, E \text{ constant} \tag{1.1}$$

is a well-known classic example of an integrable nonlinear evolution equation. Because of its many and varied applications [1–3], many exact solutions have been constructed. Those closed-form solutions that were known up to 1971 were tabulated by Benton and Platzman [4].

The application that we consider here is that of vertical water flow in unsaturated soils. In a homogeneous unsaturated soil, the one-dimensional Richards equation for volumetric water content θ is

$$\theta_t = \partial_z \left[D(\theta)\, \theta_z\right] - K'(\theta)\, \theta_z \tag{1.2}$$

(see, for example, [5]). The Burgers' equation is a particular case with D constant and $K(\theta)$ quadratic. Unlike the solutions on the whole line tabulated by Benton and Platzman [4], in the present context, solutions are applicable only on the half-line $z > 0$, representing the soil subsurface. Boundary conditions at $z = 0$ are directly relevant. The solution with constant flux boundary condition, derived by Clothier *et al.* [6], represents infiltration during steady rainfall. The solution with linearly increasing flux at $z = 0$, was used by Broadbridge and White [7] to investigate the onset of ponding under an increasing rainfall rate. Realizing the practical importance of the semi-infinite or finite domain, Calogero and de Lillo [8] developed a systematic approach to solving Burgers' equation with time-dependent boundary conditions.

The Richards equation (1.2) is often considered as a starting point of unsaturated flow theory. In reality, soils are heterogeneous and they often exhibit distinct layers. As pointed out recently by Smith [9], there are few known exact solutions to transient flow in layered media and most of these use a simple piston flow approximation, in which the water content profile is approximated by a step function. Srivastava and Yeh [10] obtained some solutions for the linear model (D and $K'(\theta)$ both piecewise constant) in two finite layers. Here, we consider a very deep lower layer. It is observed that for a range of boundary conditions, including constant flux or constant concentration boundary conditions, a travelling wave develops at large times. It is the concavity

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX of $K(\theta)$ that ensures the stability of this travelling wave solution [11,12]. For this reason, we replace the linear model by the Burgers equation but with the parameters D and E of (1.1) different in the two layers. With some extra restrictions on the soil hydraulic parameters, we are able to obtain exact solutions. These exhibit the development of the travelling wave solution in the lower layer, with the adjoining upper layer tending to a non-trivial steady state.

2. BURGERS' EQUATION ADAPTED TO LAYERED SOILS

The applicability of Burgers' equation to a homogeneous soil has been discussed by Clothier *et al.* [6]. Here we adopt a similar notation, except that indices (i) refer to the i^{th} layer, with $i = 1$ representing the upper layer and $i = 2$ the lower layer. In layer i, we normalize the volumetric water content

$$\Theta_{(i)} = \frac{\theta - \theta_n^{(i)}}{\theta_s^{(i)} - \theta_n^{(i)}}. \tag{2.1}$$

$\theta_s^{(i)}$ is the water content at saturation and $\theta_n^{(i)}$ is some low initial background level. The soil water diffusivity is assumed to be

$$D = D^{(i)}, \qquad \text{constant in each layer} \tag{2.2}$$

and the hydraulic conductivity is assumed to be

$$K = K_s^{(i)}\,\Theta_{(i)}^2, \tag{2.3}$$

where $K_s^{(i)}$ is the saturated conductivity in layer i.

In our consideration of boundary conditions at the interface between layers, an important quantity to be considered is the soil-water interaction potential Ψ. This is the potential energy per unit weight of water, due to soil-water interaction, and it must be continuous (e.g., [9]). K, D and Ψ are related (e.g., [5]) through

$$D = K\,\frac{d\Psi}{d\theta}, \qquad \Psi(\theta_s) = 0.$$

From (2.1)–(2.3), this yields

$$\Psi^{(i)} = \frac{D^{(i)}\left[\theta_s^{(i)} - \theta_n^{(i)}\right]}{K_s^{(i)}}\left[1 - \Theta_{(i)}^{-1}\right]. \tag{2.4}$$

We shall use length and time scales intrinsic to layer 1. The sorptive length scale [13] is $1/\alpha_{(1)}$ where

$$\begin{aligned}\frac{1}{\alpha_{(i)}} &= \frac{\int_{-\infty}^{0} K^{(i)}\,d\Psi}{K_s^{(i)}}\\ &= \frac{\left[\theta_s^{(i)} - \theta_n^{(i)}\right] D^{(i)}}{K_s^{(i)}}.\end{aligned} \tag{2.5}$$

The gravity time scale, differing from that of Philip [5] only by a factor of $\pi/4$, is

$$t_s = \frac{1}{\alpha_{(1)}U_{(1)}}, \qquad \text{where} \qquad U_{(i)} = \frac{K_s^{(i)}}{\theta_s^{(i)} - \theta_n^{(i)}}. \tag{2.6}$$

$U_{(i)}$ may be interpreted as the speed of the large-time travelling wave profile in a single layer of type i, saturated at the surface [11].

In terms of dimensionless variables $Z = \alpha z$, $T = t/t_s$ and Θ, the governing equation (1.2) becomes

$$\frac{\partial\Theta_{(1)}}{\partial T} = \frac{\partial^2\Theta_{(1)}}{\partial Z^2} - 2\Theta_{(1)}\frac{\partial\Theta_{(1)}}{\partial Z} \qquad \text{and} \tag{2.7a}$$

$$\frac{\partial\Theta_{(2)}}{\partial T} = \mathcal{D}\,\frac{\partial^2\Theta_{(2)}}{\partial Z^2} - 2\mathcal{U}\Theta_{(2)}\frac{\partial\Theta_{(2)}}{\partial Z}, \tag{2.7b}$$

where $\mathcal{D} = D^{(2)}/D^{(1)}$ and $\mathcal{U} = U^{(2)}/U^{(1)}$.

Here, we assume the initial condition

$$\Theta_{(i)} = 0 \tag{2.8}$$

and the constant flux boundary condition, representing a steady rainfall rate R of dimensions LT^{-1}:

$$\Theta_{(1)}^2 - \frac{\partial \Theta_{(1)}}{\partial Z} = \mathcal{R} = \frac{R}{K_s^{(1)}} \qquad \text{at } Z = 0. \tag{2.9}$$

At the two-layer interface $Z = L$, there must be continuity of potential Ψ of (2.4),

$$\lim_{Z\to L-} \frac{D^{(1)}\left[\theta_s^{(1)} - \theta_n^{(1)}\right]}{K_s^{(1)}} \left[1 - \Theta_{(1)}^{-1}\right] = \lim_{Z\to L+} \frac{D^{(2)}\left[\theta_s^{(2)} - \theta_n^{(2)}\right]}{K_s^{(2)}} \left[1 - \Theta_{(2)}^{-1}\right] \tag{2.10}$$

and continuity of water flux,

$$\lim_{Z\to L-} \Theta_{(1)}^2 - \frac{\partial \Theta_{(1)}}{\partial Z} = \lim_{Z\to L+} \mathcal{U}\Theta_{(2)}^2 - \mathcal{D}\frac{\partial \Theta_{(2)}}{\partial Z}. \tag{2.11}$$

We are required to solve the boundary value problem (2.7)–(2.11).

3. EXACT SOLUTION OF THE NONLINEAR BOUNDARY VALUE PROBLEM

In each layer, we now apply the Cole-Hopf transformation [14,15]:

$$\Theta_{(1)} = -\frac{u_Z^{(1)}}{u^{(1)}}, \tag{3.1a}$$

$$\Theta_{(2)} = -\frac{u_Z^{(2)}}{u^{(2)}}\frac{\mathcal{D}}{\mathcal{U}}. \tag{3.1b}$$

Then, (2.7) is equivalent to

$$\left[u_Z^{(1)} - u^{(1)}\frac{\partial}{\partial Z}\right]\left[u_T^{(1)} - u_{ZZ}^{(1)}\right] = 0 \qquad \text{and} \tag{3.2a}$$

$$\left[u_Z^{(2)} - u^{(2)}\frac{\partial}{\partial Z}\right]\left[u_T^{(2)} - \mathcal{D}u_{ZZ}^{(2)}\right] = 0. \tag{3.2b}$$

Therefore, it is sufficient that $u^{(1)}$ and $u^{(2)}$ obey the linear diffusion equations

$$u_T^{(1)} = u_{ZZ}^{(1)} \qquad \text{and} \tag{3.3a}$$

$$u_T^{(2)} = \mathcal{D}u_{ZZ}^{(2)}. \tag{3.3b}$$

Continuity of potential (2.10) implies

$$\begin{aligned} &\alpha_{(2)} \lim_{Z\to L-} u_Z^{(1)} \lim_{Z\to L+} u_Z^{(2)} + \alpha_{(2)} \lim_{Z\to L-} u^{(1)} \lim_{Z\to L+} u_Z^{(2)} \\ &\quad = \alpha_{(1)} \lim_{Z\to L-} u_Z^{(1)} \lim_{Z\to L+} u_Z^{(2)} + \alpha_{(1)}\frac{\mathcal{U}}{\mathcal{D}} \lim_{Z\to L-} u_Z^{(1)} \lim_{Z\to L+} u^{(2)}. \end{aligned} \tag{3.4}$$

Continuity of flux (2.11) implies

$$\lim_{Z\to L-} \frac{u_T^{(1)}}{u^{(1)}} = \lim_{Z\to L+} \frac{\mathcal{D}}{\mathcal{U}}\frac{u_T^{(2)}}{u^{(2)}}. \tag{3.5}$$

Now, we notice that the factor $\mathcal{D}/\mathcal{U}$ in (3.4) and (3.5) is identical to $\alpha_{(1)}/\alpha_{(2)}$. Therefore, in the special case that $\alpha_{(1)} = \alpha_{(2)}$, the boundary conditions (3.4) and (3.5) are both guaranteed by requiring continuity of both u and u_Z across the interface:

$$\lim_{Z\to L^-} u^{(1)} = \lim_{Z\to L^+} u^{(2)}, \tag{3.6a}$$

$$\lim_{Z\to L^-} u_Z^{(1)} = \lim_{Z\to L^+} u_Z^{(2)}. \tag{3.6b}$$

As in the case of the linear model [10], we cannot yet solve the layered medium problem except when $\alpha_{(1)} = \alpha_{(2)}$. If the water content range is uniform $(\theta_s^{(1)} - \theta_n^{(1)} = \theta_s^{(2)} - \theta_n^{(2)})$, $\alpha_{(1)} = \alpha_{(2)}$ only if diffusivity D and saturated conductivity K_s vary across the interface by the same ratio. This is unlikely to happen in practice but at least an increase in K_s is correlated with an increase in D, as to be expected.

The constant-flux boundary condition (2.9) reduces to

$$u_T^{(1)} = \mathcal{R}u^{(1)} \qquad \text{at } Z = 0.$$

Since $u^{(1)}$ is defined freely up to a scaling gauge transformation, we take

$$u^{(1)} = e^{\mathcal{R}T}, \qquad Z = 0. \tag{3.7}$$

The initial condition $\Theta_{(i)} = 0$ implies

$$u^{(i)} = \text{ constant} \qquad \text{at } T = 0.$$

To ensure compatibility with (3.6) and (3.7), we take

$$u^{(i)} = 1, \qquad T = 0. \tag{3.8}$$

The simplified linear boundary value problem that we now need to solve is

$$\frac{\partial u^{(1)}}{\partial T} = \frac{\partial^2 u^{(1)}}{\partial Z^2}, \qquad 0 < Z < L \tag{3.3a}$$

$$\frac{\partial u^{(2)}}{\partial T} = \mathcal{D}\frac{\partial^2 u^{(2)}}{\partial Z^2}, \qquad Z > L \tag{3.3b}$$

$$u^{(1)} = e^{\mathcal{R}T}, \qquad Z = 0 \tag{3.7}$$

$$\lim_{Z\to L^+} u^{(2)}(Z,T) = \lim_{Z\to L^-} u^{(1)}(Z,T) \tag{3.6a}$$

$$\lim_{Z\to L^+} u_Z^{(2)}(Z,T) = \lim_{Z\to L^-} u_Z^{(1)}(Z,T) \tag{3.6b}$$

$$u^{(1)}(Z,T) = u^{(2)}(Z,T) = 1, \qquad T = 0 \tag{3.8}$$

$$u^{(2)}(Z,T) \to 1, \qquad Z \to \infty. \tag{3.9}$$

After taking the Laplace transform $u^{(i)}(Z,T) \to \tilde{u}^{(i)}(Z,p)$, (3.3) implies

$$\tilde{u}^{(1)} = \frac{1}{p} + Ae^{-p^{\frac{1}{2}}Z} + Be^{p^{\frac{1}{2}}Z}, \qquad 0 \le Z \le L \quad \text{and} \tag{3.10a}$$

$$\tilde{u}^{(2)} = \frac{1}{p} + Ce^{-Z\sqrt{p/\mathcal{D}}} + Ee^{Z\sqrt{p/\mathcal{D}}}, \qquad Z \ge L. \tag{3.10b}$$

Boundary condition (3.9) immediately yields $E = 0$. The remaining boundary conditions (3.6a), (3.6b) and (3.7) transform to a system of three algebraic equations for A, B and C:

$$Ae^{-L\sqrt{p}} + \left[\frac{1}{p-\mathcal{R}} - \frac{1}{p} - A\right] e^{L\sqrt{p}} = Ce^{-L\sqrt{p/\mathcal{D}}}, \tag{3.11}$$

$$-Ae^{-L\sqrt{p}} + \left[\frac{1}{p-\mathcal{R}} - \frac{1}{p} - A\right] e^{L\sqrt{p}} = -\frac{1}{\sqrt{\mathcal{D}}}Ce^{-L\sqrt{p/\mathcal{D}}}, \tag{3.12}$$

$$B = \frac{1}{p-\mathcal{R}} - \frac{1}{p} - A. \tag{3.13}$$

We write the unique solution in a series form that will allow inversion of the Laplace transform from standard tables:

$$A = \left[\frac{1}{p-\mathcal{R}} - \frac{1}{p}\right] \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sqrt{\mathcal{D}}-1}{\sqrt{\mathcal{D}}+1}\right)^n e^{-2nL\sqrt{p}}, \tag{3.14}$$

$$B = \left[\frac{1}{p} - \frac{1}{p-\mathcal{R}}\right] \sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{\mathcal{D}}-1}{\sqrt{\mathcal{D}}+1}\right)^n e^{-2nL\sqrt{p}}, \tag{3.15}$$

$$C = \left[\frac{1}{p-\mathcal{R}} - \frac{1}{p}\right] \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sqrt{\mathcal{D}}-1}{\sqrt{\mathcal{D}}+1}\right)^n \left[e^{-(2n+1-1/\sqrt{\mathcal{D}})L\sqrt{p}} - e^{-(2n-1-1/\sqrt{\mathcal{D}})L\sqrt{p}}\right]$$
$$+ \left[\frac{1}{p-\mathcal{R}} - \frac{1}{p}\right] e^{(1+1/\sqrt{\mathcal{D}})L\sqrt{p}}. \tag{3.16}$$

Inverting (3.10), we obtain

$$u^{(1)} = 1 + \frac{1}{2} e^{\mathcal{R}T} \left\{ e^{-Z\sqrt{\mathcal{R}}} \operatorname{erfc}\left[\frac{Z}{2\sqrt{T}} - \sqrt{\mathcal{R}T}\right] + e^{Z\sqrt{\mathcal{R}}} \operatorname{erfc}\left[\frac{Z}{2\sqrt{T}} + \sqrt{\mathcal{R}T}\right]\right\} - \operatorname{erfc}\left(\frac{Z}{2\sqrt{T}}\right)$$
$$+ \sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{\mathcal{D}}-1}{\sqrt{\mathcal{D}}+1}\right)^n \left\{ \frac{1}{2} e^{\mathcal{R}T} \left[e^{-\sqrt{\mathcal{R}}[2nL+Z]} \operatorname{erfc}\left(\frac{2nL+Z}{2\sqrt{T}} - \sqrt{\mathcal{R}T}\right)\right.\right.$$
$$+ e^{\sqrt{\mathcal{R}}[2nL+Z]} \operatorname{erfc}\left(\frac{2nL+Z}{2\sqrt{T}} + \sqrt{\mathcal{R}T}\right) - e^{-\sqrt{\mathcal{R}}[2nL-Z]} \operatorname{erfc}\left(\frac{2nL-Z}{2\sqrt{T}} - \sqrt{\mathcal{R}T}\right)$$
$$\left.\left. - e^{\sqrt{\mathcal{R}}[2nL-Z]} \operatorname{erfc}\left(\frac{2nL-Z}{2\sqrt{T}} + \sqrt{\mathcal{R}T}\right)\right] + \operatorname{erfc}\frac{2nL-Z}{2\sqrt{T}} - \operatorname{erfc}\frac{2nL+Z}{2\sqrt{T}}\right\}, \tag{3.17a}$$

$$u^{(2)} = 1 + \frac{1}{2} e^{\mathcal{R}T} \left\{ e^{-Z_1\mathcal{R}} \operatorname{erfc}\left[\frac{Z_1}{2\sqrt{T}} - \sqrt{\mathcal{R}T}\right] + e^{Z_1\mathcal{R}} \operatorname{erfc}\left[\frac{Z_1}{2\sqrt{T}} + \sqrt{\mathcal{R}T}\right]\right\} - \operatorname{erfc}\left(\frac{Z_1}{2\sqrt{T}}\right)$$
$$+ \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sqrt{\mathcal{D}}-1}{\sqrt{\mathcal{D}}+1}\right)^n \left\{ \frac{1}{2} e^{\mathcal{R}T} \left[e^{-\sqrt{\mathcal{R}}[2(n+1)L+Z_1]} \operatorname{erfc}\left(\frac{2[n+1]L+Z_1}{2\sqrt{T}} - \sqrt{\mathcal{R}T}\right)\right.\right.$$
$$+ e^{\sqrt{\mathcal{R}}[2(n+1)L+Z_1]} \operatorname{erfc}\left(\frac{2[n+1]L+Z_1}{2\sqrt{T}} + \sqrt{\mathcal{R}T}\right)$$
$$- e^{-\sqrt{\mathcal{R}}[2nL+Z_1]} \operatorname{erfc}\left(\frac{2nL+Z_1}{2\sqrt{T}} - \sqrt{\mathcal{R}T}\right)$$
$$\left. - e^{\sqrt{\mathcal{R}}[2nL+Z_1]} \operatorname{erfc}\left(\frac{2nL+Z_1}{2\sqrt{T}} + \sqrt{\mathcal{R}T}\right)\right]$$
$$\left. + \operatorname{erfc}\frac{2nL+Z_1}{2\sqrt{T}} - \operatorname{erfc}\frac{2(n+1)L+Z_1}{2\sqrt{T}}\right\},$$

where

$$Z_1 = \frac{(Z-L)}{\sqrt{\mathcal{D}}} - L. \tag{3.17b}$$

Hence, via (3.1), we have a closed-form solution to the nonlinear boundary value problem (2.7)–(2.11) for unsaturated flow in layered media. Although we have concentrated on the constant-flux boundary condition at $Z = 0$, we expect that our method may also be applied to other boundary conditions, such as prescribed concentration or prescribed time-dependent flux.

4. DISCUSSION OF RESULTS

The analytic solution contains much interesting structure, which we now illustrate. In Figure 1, we display the developing concentration profile for a dimensionless rainfall rate $\mathcal{R} = 1$, when the upper layer 1, of dimensionless depth $L = 2$, has half the conductivity of layer 2 ($\mathcal{D} = 2$).

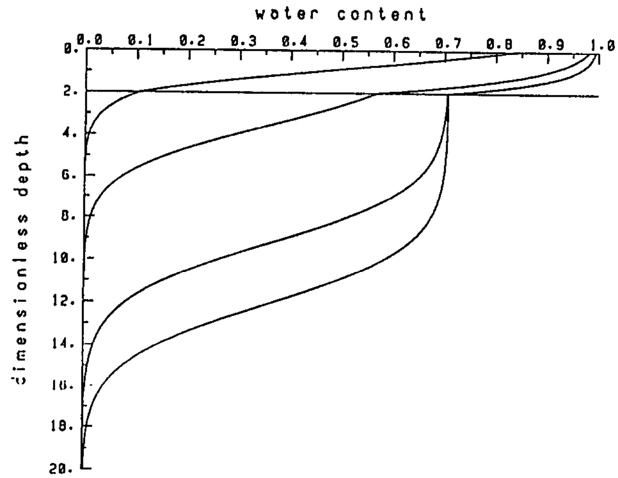

Figure 1. Analytic solution for Burgers' equation in a resistive layer overlaying a more conductive medium. Surface flux is $\mathcal{R} = 1.0$, layer interface is at depth $L = 2.0$ and ratio of diffusivity (and of conductivity) in layer 2 to that in layer 1 is $\mathcal{D} = 2.0$. Output times are given by dimensionless cumulative infiltration $RT = 1.0$, 3.0, 7.0 and 9.0.

The normalized water content Θ is continuous at the interface $Z = L$. However, this may represent a discontinuous volumetric water content θ if the parameters $\theta_n^{(i)}$ and $\theta_s^{(i)}$ vary from the top to the bottom layer.

In order to maintain a continuous water flux across the interface, the gradient $\frac{\partial\Theta}{\partial Z}$ must be discontinuous. Noting that $\mathcal{D} = \mathcal{U}$ here, equation (2.11), for continuity of water flux, implies that at $Z = L$,

$$\frac{\partial\Theta_{(1)}}{\partial Z} - \frac{\partial\Theta_{(2)}}{\partial Z} = (1 - \mathcal{D})\left(\Theta^2 - \frac{\partial\Theta_{(2)}}{\partial Z}\right), \tag{4.1}$$

where Θ is the common value of $\Theta_{(1)}$ and $\Theta_{(2)}$. The factor $\Theta^2 - \frac{\partial\Theta_{(2)}}{\partial Z}$ in (4.1) is the dimensionless water flux which is everywhere positive. In the case $\mathcal{D} > 1$, $\frac{\partial\Theta}{\partial Z} < 0$, and (4.1) implies that at $Z = L$,

$$\left|\frac{\partial\Theta_{(1)}}{\partial Z}\right| > \left|\frac{\partial\Theta_{(2)}}{\partial Z}\right|. \tag{4.2}$$

In this case, water content profiles are steeper in layer 1 than in layer 2.

At the later times $T = 7.0$ and $T = 9.0$ depicted in Figure 1, the familiar travelling wave solution to Burgers' equation is already evident in layer 2. At these two times, there is no discernible change to the water content profile in layer 1. The latter must be very close to its steady state and the flux must be almost uniform throughout layer 1. At $Z = L$, the dimensionless flux must be close to $\mathcal{R}$, the value at $Z = 0$. As time progresses, water content gradients at the top of layer 2 approach zero. As $T \to \infty$, at $Z = L^+$,

$$\mathcal{D}\left(\Theta_{(2)}^2 - \frac{\partial\Theta_{(2)}}{\partial Z}\right) \to \mathcal{R} \quad \text{and} \quad \frac{\partial\Theta_{(2)}}{\partial Z} \to 0,$$
$$\text{so } \Theta_{(2)} \to \sqrt{\frac{\mathcal{R}}{\mathcal{D}}}. \tag{4.3}$$

In the case of Figure 1, $\Theta_{(2)}$ approaches $\sqrt{2}/2$. Due to the presence of layer 2, the steady state in layer 1 is not trivial. If layer 1 were infinite in extent, $\Theta_{(1)}$ would approach $\sqrt{\mathcal{R}}$ pointwise as $T \to \infty$. However, as shown in Figure 1, when layer 2 is present, there is a persistent water content gradient in layer 1, with a corresponding persistent diffusive component to the flux, so that the limiting value of $\Theta_{(1)}$ is everywhere less than $\sqrt{\mathcal{R}}$ (= 1 in the case of Figure 1).

In Figure 2, we illustrate the evolving profile when $\mathcal{R} = 0.25$ and the upper layer has four times the conductivity of the lower layer ($\mathcal{D} = 0.25$).

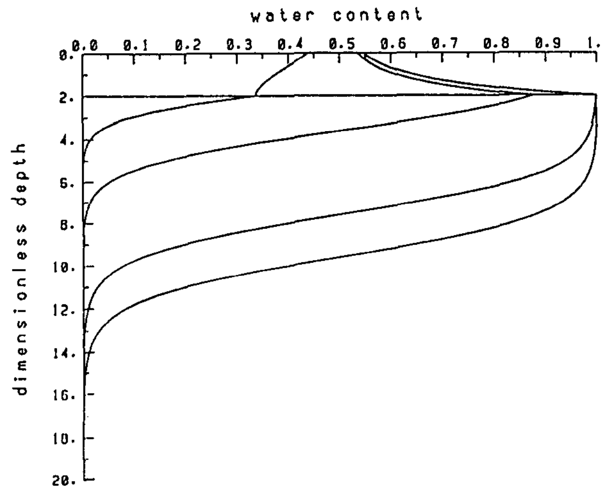

Figure 2. Analytic solution for Burgers' equation in a conductive layer overlaying a more resistive medium. $\mathcal{R} = 0.25$, $L = 2.0$, and $\mathcal{D} = 0.25$. Output times are given by the same values of RT as in Figure 1.

By (4.3), $\Theta_{(2)}$ approaches 1, as seen in the profiles with $RT = 7.0$ and $RT = 9.0$ when the travelling wave profile is evident in layer 2. By (4.1), $\frac{\partial\Theta_{(1)}}{\partial Z} > \frac{\partial\Theta_{(2)}}{\partial Z}$. At the early time $T = 1.0$, $\frac{\partial\Theta_{(1)}}{\partial Z}$ and $\frac{\partial\Theta_{(2)}}{\partial Z}$ are both negative, with a steeper gradient in layer 2,

$$\left|\frac{\partial\Theta_{(2)}}{\partial Z}\right| > \left|\frac{\partial\Theta_{(1)}}{\partial Z}\right|.$$

However, at later times, due to the higher resistance to flow in layer 2, water builds up at the interface and $\frac{\partial\Theta_{(1)}}{\partial Z}$ is positive while $\frac{\partial\Theta_{(2)}}{\partial Z}$ is negative. The sharp change in character of the concentration profile at a layer interface, and the development of a travelling wave profile at depth, have now both been captured in an analytic solution.

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