One-Dimensional Saturated Flow with Random Saturated Hydraulic Conductivity Parameter

Consider one-dimensional, steady-state flow through a heterogeneous medium under fully saturated conditions. The governing equation for the flow process is given as:

$$\frac{d}{dx}\left[K(x)\frac{d\phi}{dx}\right]=0 \quad \textbf{(1)}$$

where ϕ is the total head, x is the horizontal coordinate, and $K(x)$ is the saturated hydraulic conductivity, which is a function of location. The governing equation can be written as

$$\frac{d\ln K}{dx}\frac{d\phi}{dx}+\frac{d^2\phi}{dx^2}=0 \quad \textbf{(2)}$$

If the natural logarithm of K, lnK, and ϕ are assumed to be a statistically homogeneous stochastic process, they then can be decomposed to mean and perturbation components,

$$\ln K = F + f \qquad \phi = H + h \quad \textbf{(3)}$$

where $E[\ln K] = F$ and $E[f] = 0$ and $E[\phi] = H$ and $E[h] = 0$. Substituting (3) into (2), and neglecting the products of the perturbations, taking the expected value, and assuming the mean saturated hydraulic conductivity is independent of x, the mean flow equation, after neglecting second order terms, is given as

$$\frac{d^2H}{dx^2}\approx 0 \quad \textbf{(4)}$$

The governing flow equation in terms of the perturbations becomes

$$\frac{d^2h}{dx^2}+J\frac{df}{dx}\approx 0 \quad \textbf{(5)}$$

where J is -dH/dx. An application of Fourier-Stieltjes representation for the f, and h,

$$f=\int_{-\infty}^{\infty}e^{ikx}dz_f(k) \qquad h=\int_{-\infty}^{\infty}e^{ikx}dz_h(k) \quad \textbf{(6)}$$

leads to the Fourier amplitude relations of h, and f stochastic precesses:

$$dZ_h = \frac{iJdZ_f}{k} \qquad \textbf{(7)}$$

Furthermore, the spectral density function of the head process is given by

$$S_{hh}(k) = \frac{J^2 S_{ff}(k)}{k^2} \qquad \textbf{(8)}$$

where S_{hh} and S_{ff} are the spectra of the h and f processes. In order to evaluate the head spectrum, S_{hh}, knowledge of spectrum of f processes is necessary. To avoid a singularity in (8) when k=0, a hole covariance function is chosen to derive the spectrum for *f*.

$$R_{ff}(\xi) = \sigma_f^2 (1 - |\xi|/\ell) \exp(-|\xi|/\ell) \qquad \textbf{(9)}$$

where ξ is the separation distance, ℓ is the correlation scale at which $R_{ff}(\xi)$. The corresponding spectrum is obtained by

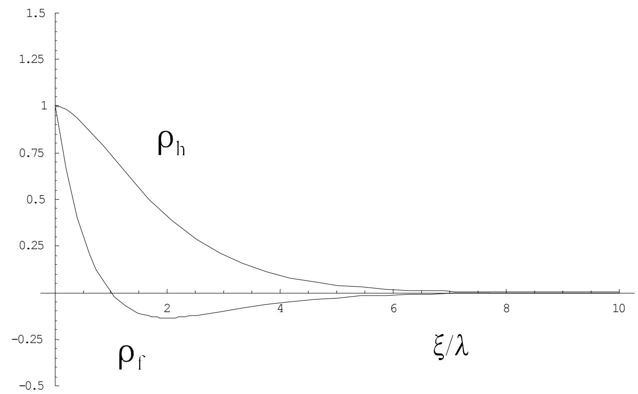
$$S_{ff}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\xi} R_{ff}(\xi) d\xi = \frac{2k^2 \sigma_f^2 \ell^3}{\pi(1 + k^2 \ell^2)^2} \qquad \textbf{(10)}$$

Using this spectrum function for f in (8), the covariance of h can be derived as

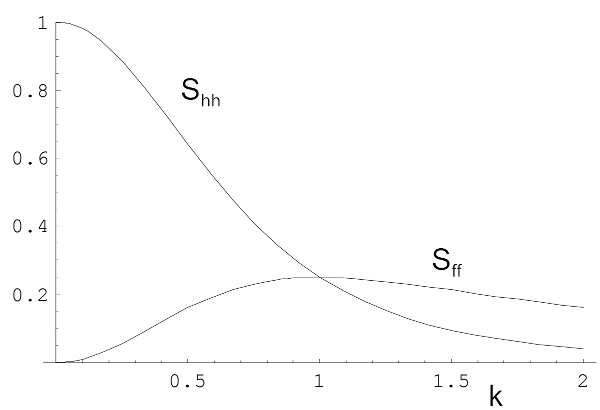
$$S_{hh}(k) = \int_{-\infty}^{\infty} e^{ik\xi} S_{ff}(k) dk = \int_{-\infty}^{\infty} \frac{e^{ik\xi} 2J^2 \sigma_f^2 \ell^3}{\pi(1 + k^2\ell^2)^2} dk = J^2 \sigma_f^2 \ell^2 (1 + |\xi|/\ell) \exp(-|\xi|/\ell) \qquad \textbf{(11)}$$

and the variance of h is

$$\sigma_h^2 = R_{hh}(0) = \int_{-\infty}^{\infty} S_{hh}(k) dk = J^2 \sigma_f^2 \ell^2 \qquad \textbf{(12)}$$



Plots of head and conductivity correlation functions, ρ_h and ρ_f, respectively, are shown in the figure. They indicate that head perturbations are correlated over a longer distance than conductivity perturbations.



The above figure shows normalized head and conductivity spectra. It indicates that high-wave number head spectrum is attenuated, indicating the head field is smooth due to a filtering effect.

HOMEWORK:

Use a hole function as the covariance function of the lnK process to derive the cross-covariance function of head and lnK process and graph the function as a function of separation distance. That is, show

$$R_{fh}(\xi) = \sigma_f^2 J\ell(-\frac{\xi}{\ell})\exp(-\frac{\xi}{\ell}) \quad for \quad \xi \geq 0$$

and

$$R_{fh}(\xi) = -\sigma_f^2 J\ell(-\frac{|\xi|}{\ell})\exp(-\frac{|\xi|}{\ell}) \quad for \quad \xi < 0$$

Please discuss the physical meaning of the graph

General 3-D Saturated Flow.

The general three-dimensional steady flow equation can be written as

$$\frac{\partial}{\partial x_i}\left(K\frac{\partial \phi}{\partial x_i}\right)=0 \qquad i=1,2,3 \tag{2}$$

where the hydraulic conductivity, K, is assumed locally isotropic. The Einstein summation convention is used. Expanding and dividing by nonzero conductivity,

$$\frac{\partial^2 \phi}{\partial x_i^2}+\frac{\partial \ln K}{\partial x_i}\frac{\partial \phi}{\partial x_i}=0 \tag{3}$$

If the variables are expressed in terms of means and perturbations,

$$\phi = H + h \qquad E[\phi]=H \qquad E[H]=0 \tag{4a}$$

$$\ln K = F + f \quad E[\ln K]=F \quad E[f]=0 \tag{4c}$$

Substituting (4) and (5) into (3), the mean flow equation is

$$\frac{\partial^2 H}{\partial x_i^2}+\frac{\partial F}{\partial x_1}+\frac{\partial F}{\partial x_i}\frac{\partial H}{\partial x_i}+E\left[\frac{\partial f \partial h}{\partial x_i \partial x_i}\right]=0 \tag{6}$$

Substituting (6) from (3) with F assumed to be constant and neglecting products of perturbation quantities, we have

$$\frac{\partial^2 h}{\partial x_i^2}-J_i\frac{\partial f}{\partial x_i}\approx 0 \tag{7}$$

where $J_i = -\partial\phi/\partial x_i$ is the hydraulic gradient. Next, Fourier-Stieljes integral representations [*Lumley and Panofsky*, 1964] are used for the random processes *h* and *f*, that is,

$$h(x_1, x_2, x_3) = \int_{-\infty}^{\infty} e^{ik \cdot x} dZ_h(k)$$

and

$$f(x_1, x_2, x_3) = \int_{-\infty}^{\infty} e^{ik \cdot x} dZ_f(k)$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the position vector, and $\mathbf{k} = (k_1, k_2, k_3)$ is the wave number vector. After substitution and manipulation, the expression relating the complex Fourier amplitudes of h and f fluctuations is

$$dZ_h = \frac{iJ_i k_i}{k^2} dZ_f \tag{9}$$

Finally, multiplying both sides of (9) by the complex conjugate of the Fourier amplitude dZ_h, taking mean values, and using the spectral representation theorem produces the spectral relationship

$$S_{hh} = \frac{(J_i k_i)^2}{k^4} S_{ff} \tag{10}$$

Equation (10) is the spectral solution to the stochastic partial differential equations governing the steady state three-dimensional flow in saturated porous media. Particular solutions, which depend on the type of covariance function used to describe the heterogeneity of the medium, will be evaluated in the next section.

Generalized Effective Hydraulic Conductivity Relation

The general form of effective hydraulic conductivity can be derived from the following specific discharge equation. Assuming local isotropy of the hydraulic conductivity, the Darcy equation for a three-dimensional flow becomes

$$q_i = -K \frac{\partial \phi}{\partial x_i} = K_m \left[1 + f + \frac{f^2}{2} + \ldots \right] \left[J_1 + \frac{\partial h}{\partial xi} \right] \tag{11}$$

where $i = 1, 2$, and 3. Taking expected values of (11), dropping terms that are beyond the second order, and noting that $E[h \partial h / \partial x_i] = (½) \partial E(h^2) / \partial x_i = 0$,

$$E[q_i] = K_m\left[\left(1+\frac{E[f^2]}{2}\right)J_i + E\left(f\frac{\partial h}{\partial x_i}\right)\right]$$
$$= K_m\left[\left(1+\frac{E[f^2]}{2}\right)\delta_{ij} + F_{ij}\right]J_j \qquad (12)$$
$$= \overline{K}_{ij}J_j$$

where $E[f\,\partial h/\partial x_i] = F_{ij}J_j$, and δ_{ij} is the Kronecker delta. Thus, even though local isotropy is assumed in the derivation of (12), the resulting equation produces a tensorial form of the mean Darcy equation, where $\overline{K}_{ij}$ is the effective hydraulic conductivity tensor.

Head Variance in Three-dimensional Media

To evaluate head variance, a three-dimensional covariance function for lnK must be specified. In this case, an isotropic exponential covariance function is used:

$$R_{ff}(\vec{\xi}) = \sigma_f^2 \exp(-|\vec{\xi}|/\lambda)$$

where ξ is the separation vector and λ is the integral scale. The corresponding spectral density function is given as

$$S_{ff}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{\xi}} R_{ff}(\vec{\xi})d\vec{\xi} = \frac{\sigma_f^2\lambda^3}{\pi^2(1+k^2\lambda^2)^2}$$

Using this spectral density function, the head variance of steady-state, uniform flow (J_1=J≠0, and J_2=J_3=0) in three-dimensional heterogeneous media becomes

$$\sigma_h^2 = R_{hh}(0) = \int_{-\infty}^{\infty} S_{hh}(k)dk = \frac{J^2\sigma_f^2\lambda^2}{3}$$

To compare this with one-dimensional result, a comparable correlation scale must be used. In one-dimensional problem, a hole function was used:

$$R_{ff}(\xi) = \sigma_f^2(1-|\xi|/\ell)\exp(-|\xi|/\ell)$$

and an exponential covariance function was used for three-dimensional problem. That is,

$$R_{ff}(\vec{\xi}) = \sigma_f^2 \exp(-|\vec{\xi}|/\lambda)$$

Since ℓ/λ=5/2, the shapes of the two functions are almost identical, this relation will be used in the comparison. The head variances for one- and three-dimensional flow problems are

$$\sigma_h^2 = J^2 \sigma_f^2 \ell^2 \quad and \quad \sigma_h^2 = \frac{J^2 \sigma_f^2 \lambda^2}{3} = \frac{4 J^2 \sigma_f^2 \ell^2}{75}$$

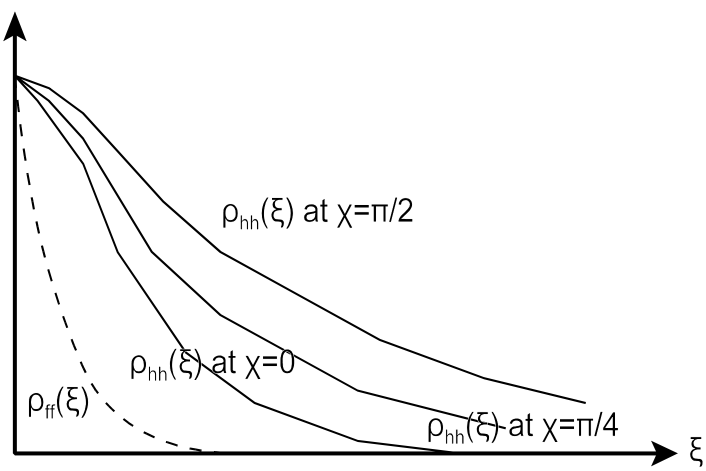
respectively. The comparison shows that the head variance of three-dimensional flow is much smaller than one-dimensional model due to multidimensional flow paths allowing flow to avoid low permeable zones.

Head Covariance Function:

The covariance of head for the 3-D model can be determined by

$$R_{hh}(\vec{\xi}) = \iiint e^{i\vec{k}\cdot\vec{\xi}} S_{hh}(\vec{k}) d\vec{k}$$

The analytical integration is complicated and please see Bakr et al., (1979) for details. However, it can always be evaluated numerically. The figure below schematically illustrated the behavior of the correlation function for head and lnK processes. The angle between the mean flow direction and the separation distance vector is denoted as χ.



Based on the figure, conclusions can be drawn: 1) head perturbations are correlated over a longer distance in the direction perpendicular to the mean flow direction than in other directions; 2) The head covariance function is always anisotropic even though the lnK covariance is isotropic.

Effective Hydraulic Conductivity:

Formulations the effective hydraulic conductivity for 3-D stochastic heterogeneous media can be found in Gelhar and Axness (1983) or Gelhar (1993). Graphic results are shown below.

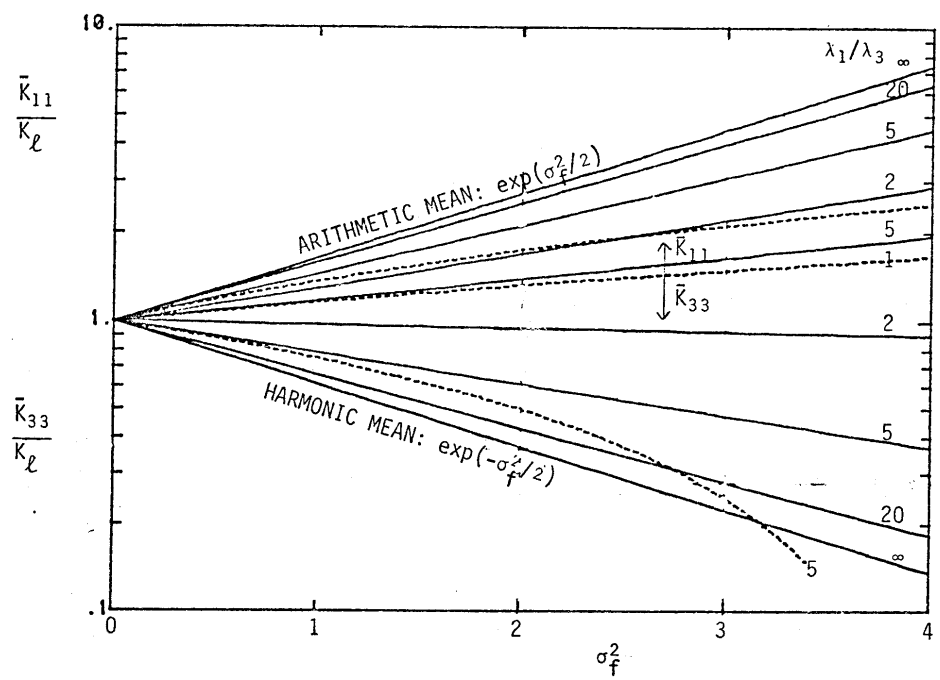


Fig. 5.3. Effective hydraulic conductivities parallel ($\bar{K}_{11}$) and perpendicular ($\bar{K}_{33}$) to bedding; the solid lines are the generalization (5.12) and the dashed lines are the first order results (5.9).

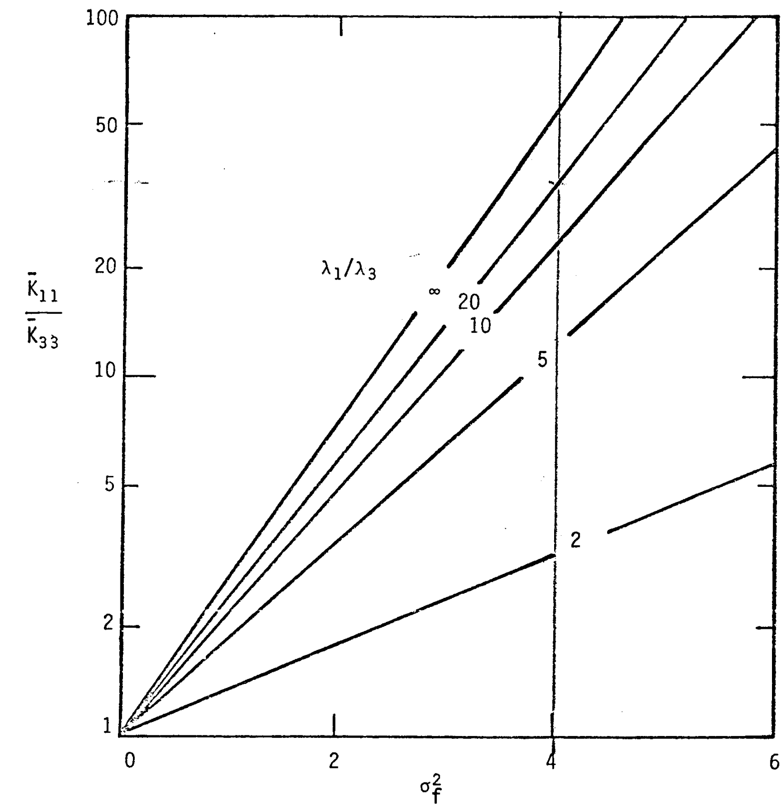


Fig. 5.4. Anisotropy of effective hydraulic conductivity as a function of the variance of lnK, σ_f^2, and aspect ratio, λ_1/λ_3, for the

Analysis of Transient Flow

The transient flow equation for depth-averaged groundwater flow is given by

$$\frac{\partial}{\partial x}(T\frac{\partial \phi}{\partial x})+\frac{\partial}{\partial y}(T\frac{\partial \phi}{\partial y})=S\frac{\partial \phi}{\partial t}$$

Rewrite it in terms of ln T:

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial \ln T}{\partial x}\frac{\partial \phi}{\partial x}+\frac{\partial \ln T}{\partial y}\frac{\partial \phi}{\partial y}=\frac{S}{T}\frac{\partial \phi}{\partial t}$$

Expressing ln T and ϕ in terms of means and perturbations:

$$\ln T = F + f, \quad \phi = H + h$$

and approximating 1/T by

$$\frac{1}{T}\approx\frac{1}{T_g}(1-f+\frac{f^2}{2})$$

The mean equation, assuming uniform mean flow in the x direction only, becomes

$$\frac{\partial^2 H}{\partial x^2}+E[\frac{\partial f}{\partial x}\frac{\partial h}{\partial x}]+E[\frac{\partial f}{\partial y}\frac{\partial h}{\partial y}]=\frac{S}{T_g}\frac{\partial H}{\partial t}(1+\frac{\sigma_f^2}{2})$$

and the perturbation equation after neglecting second- and higher-order terms becomes

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}-J_x\frac{\partial f}{\partial x}=-\frac{S}{T_g}\frac{\partial H}{\partial t}f$$

where $J_x = -\partial H/\partial x$ is the mean gradient. Applying Fourier-Stieljes representation theorem, we have a spectral relation between h and f:

$$S_{hh}(k_1,k_2)=\left[\frac{k_1^2 J_1^2}{(k_1^2+k_2^2)^2}+\frac{\left(\frac{S}{T_g}\frac{\partial H}{\partial t}\right)^2}{(k_1^2+k_2^2)^2}\right]S_{ff}(k_1,k_2)$$

Using a spectrum

$$S_{ff}(k_1,k_2)=\frac{\alpha(k_1^2+k_2^2)}{\beta^4(k_1^2+k_2^2+1/\beta^2)^4};\quad \alpha=\frac{3\sigma^2\beta^2}{\pi},\quad \beta=\frac{16\lambda}{3\pi}$$

the head variance is obtained as

$$\sigma_h^2=\left[\frac{J_1}{4}+\left(\frac{S}{T_g}\frac{\partial H}{\partial t}\frac{16\lambda}{3\pi}\right)^2\right]\left(\frac{16\lambda}{3\pi}\right)^2\sigma_f^2$$

As indicated in this formula, the head variance in a transient flow consists of two parts: 1) the variance due to the mean spatial gradient, J_1 , which varies with time and 2) the temporal mean head gradient, ∂H/∂t. Moreover, the correlation structure has greater effects on the transient head variance than the steady head variance. These observations are also true for the head covariance functions. Detailed formulations of the head covariance functions are given in Mizell et al., (198?), and Gelhar (1993).

Effective Hydraulic Conductivity

Using the above procedures, the term, E[f∂h/∂x] in (12) can be evaluated and can be used to determine the effective hydraulic conductivity under transient flow conditions. Even though analytical expression for the effective conductivity is not available here, based on the expression of the head variance, we can deduce that effective hydraulic conductivity for transient flow will depend on the mean spatial gradient and the mean temporal gradient, which are dependent on time. Consequently, the effective hydraulic conductivity will be time varying even if ergodicity assumption is true.

Head covariance and cross-covariance between h and f behaviors in 2-D horizontal steady flow problems.

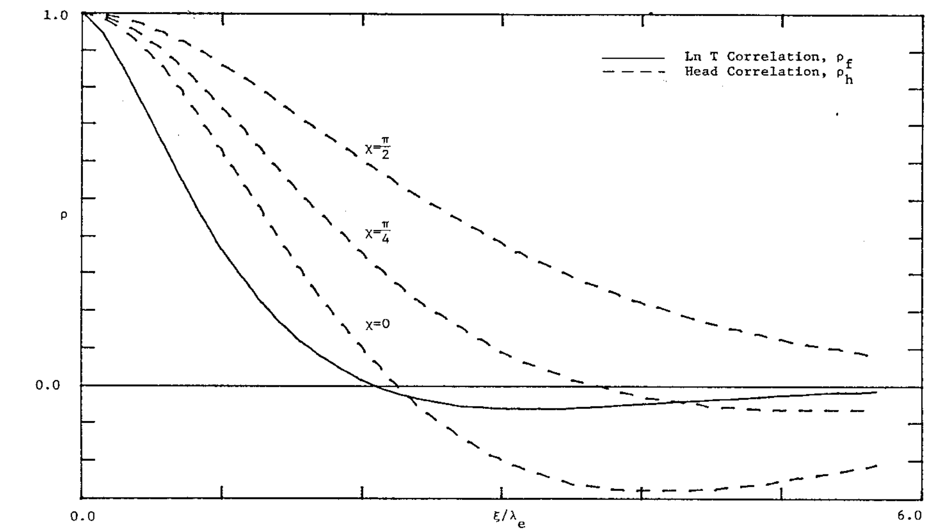


Figure 2.5: Ln T and Head Correlation Functions assuming Input Spectrum B

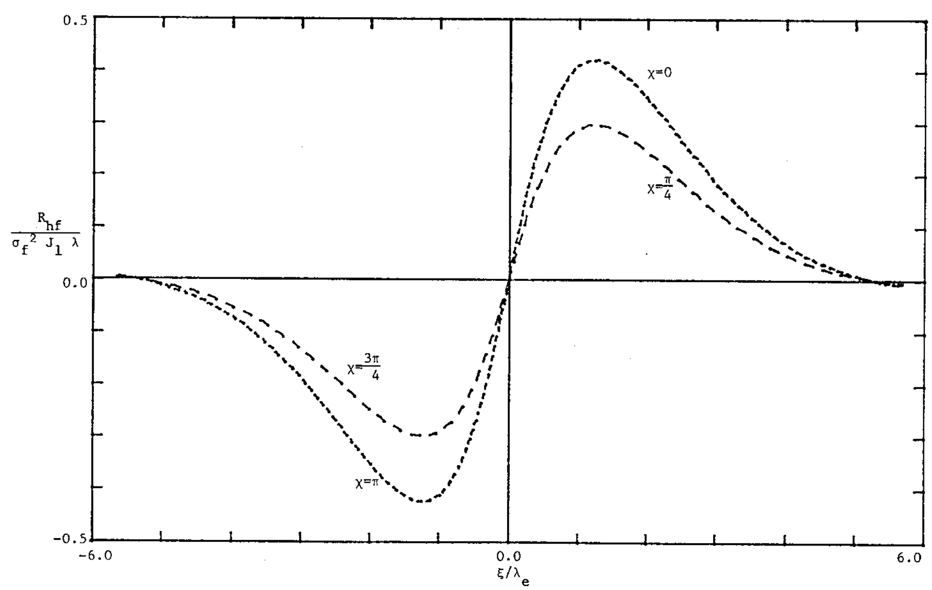


Figure 2.8: Dimensionless Cross-Covariance Function between Ln T and Head assuming Input Spectrum B

HOMEWORK

1. Apply the procedures discussed for 2-D transient flow to derive the head covariance and effective hydraulic conductivity for 1-D transient flow problems. Discuss results.

2. How does the ergodicity assumption affect the result?

3. How would the spatial variation of the storage coefficient, boundary condition, etc affect the effective hydraulic conductivity? Are these effective parameters nothing but fudge factors?

4. How would the stationarity assumption used in the above analysis affect the analysis?