

L-4. STOCHASTIC DIFFERENTIAL EQUATIONS

A stochastic differential equation is an equation, say,

$$f(X(t), X'(t), X''(t), \dots, Y(t), Y'(t), Y''(t), \dots, t) = 0,$$

where $X(t), X'(t) \dots Y(t), Y'(t) \dots$ are stochastic processes along with some boundary conditions or initial conditions.

Solution: The most general solution would be the one specifying the joint distribution of $X(t_1), X(t_2) \dots X(t_n)$ for all $t_1 \dots t_n$. Instead, we might settle for information about the moments of $X(t)$. i.e. $E[X(t)], E[X^2(t)], E[X(t)X(t+S)]$.

Types of Stochastic differential equation

(a) Random initial and boundary condition

Solution obtained as in ordinary differential equation then introduce the random initial or boundary condition.

(b) Random inhomogeneous part

$$a \frac{d^2 X}{dt^2} + b \frac{dX}{dt} + X(t) = Y(t)$$

Hydrology example

$$T \nabla^2 h(x) = \varepsilon(x)$$

where T is the transmissivity, h is head, and $\varepsilon(x)$ is recharge.

(c) Random stochastic coefficients

$$\nabla \cdot [K(x) \nabla \phi(x)] = 0$$

where $K(x)$ is a stochastic process representing a spatially varied conductivity field and $\phi(x)$ is the hydraulic head, which is also a stochastic process.

(d) Random coefficients, inhomogeneous parts, boundary and initial conditions.

Example: Random Inhomogeneous Part

Suppose that $Y(t)$ is a second-order stationary process and $V(t)$ is related to $Y(t)$ through the differential equation.

$$a_1 V(t) + a_2 \frac{dV(t)}{dt} = Y(t)$$

Suppose Y has a spectral density, $S_y(\omega)$, we want to know the spectral density of $V(t)$, and say a_1 and a_2 are constants. Let's express V and Y into means and perturbations.

$$V = E[V] + V'$$

$$Y = E[Y] + Y'$$

where $E[V'] = 0$ and $E[Y'] = 0$. Substitute them into the differential equation, we obtain

$$a_1 (E[V] + V') + a_2 \frac{d(E[V] + V')}{dt} = E[Y] + Y'$$

Taking the expected value of the equation, we have the mean equation

$$a_1 E[V] + a_2 \frac{dE[V]}{dt} = E[Y]$$

Notice that

$$E\left[\frac{dV}{dt}\right] = \frac{dE[V]}{dt}$$

Subtracting the mean equation from (1), we obtain a perturbation equation:

$$a_1 V' + a_2 \frac{dV'}{dt} = Y'$$

Now, we use the Fourier-Stieljes Representation for V and Y :

$$y(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_y(\omega)$$

$$v(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_v(\omega)$$

and substitute them into the perturbation equation, we have

$$a_1 \int_{-\infty}^{\infty} e^{i\omega t} dZ_V + ia_2 \int_{-\infty}^{\infty} e^{i\omega t} dZ_V = \int_{-\infty}^{\infty} e^{i\omega t} dZ_Y$$

After rearrangement, we have

$$\int_{-\infty}^{\infty} e^{i\omega t} (a_1 + ia_2\omega) dZ_V = \int_{-\infty}^{\infty} e^{i\omega t} dZ_Y$$

or

$$(a_1 + ia_2\omega)dZ_V = dZ_Y$$

Now multiplying it with its conjugates and taking the expect value, we have

$$|(a_1^2 + a_2^2\omega^2)|E[dZ_V dZ_V^*] = E[dZ_Y dZ_Y^*]$$

which leads to

$$|(a_1^2 + a_2^2\omega^2)|S_V(\omega)d\omega = S_Y(\omega)d\omega$$

or

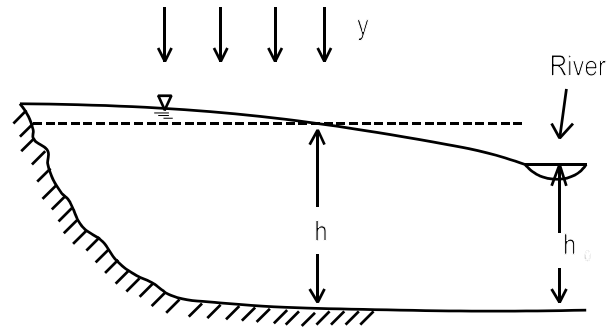
$$S_V(\omega) = \frac{1}{|(a_1^2 + a_2^2\omega^2)|} S_Y(\omega)$$

Applications in Ground Water Hydrology

Stochastic Analysis of Phreatic Aquifer, Gelhar, *WRR* 10(3), 589~545, 1974.

Lumped Parameter Model

(Gelhar & Wilson, 1974, *Ground Water*, 12(6), 399-408, Dec. 1974)



- S = specific yield
- h = average water level
- a = outflow constant
- y = recharge
- h_0 = outflow datum

The governing linear reservoir model is

$$s \frac{dh}{dt} + a(h - h_0) = y(t)$$

If y is a stationary random process, then h is a stationary random process. We express them in terms of means and perturbations

$$\begin{aligned} h &= \bar{h} + h' & E[h] &= \bar{h}, & E[h'] &= 0 \\ y &= \bar{y} + y' & E[y] &= \bar{y}, & E[y'] &= 0 \end{aligned}$$

Substitution into (1) leads to

$$s \frac{d\bar{h} + h'}{dt} + a(\bar{h} + h - h_0) = \bar{y} + y'$$

Taking the expected value yields the mean equation

$$s \frac{d\bar{h}}{dt} + a(\bar{h} - h_0) = \bar{y}$$

Perturbation equation is given as

$$s \frac{dh'}{dt} + ah' = y'$$

Now, we use Fourier-Stieljes Representation for h' and y' :

$$h'(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_h \quad y'(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_y$$

and substitute them into the perturbation equation, we have

$$s \int_{-\infty}^{\infty} \frac{d}{dt} e^{i\omega t} dZ_h(\omega) + a \int_{-\infty}^{\infty} e^{i\omega t} dZ_h(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_y(\omega)$$

or

$$\int_{-\infty}^{\infty} e^{i\omega t} [(si\omega + a)dZ_h(\omega) - dZ_y(\omega)] = 0$$

Therefore

$$S_{hh}(\omega)d\omega = E[dZ_h dZ_h^*] = E[dZ_y dZ_y^*] S_{yy}(\omega)d\omega$$

$$dZ_h = \frac{dZ_y}{(is\omega + a)}$$

$$dZ_h^* = \frac{dZ_y^*}{(-is\omega + a)}$$

$$S_{hh}(\omega) / S_{yy}(\omega) = \frac{1}{(s^2\omega^2 + a^2)}$$

Assume

$$Y = \text{recharge} = \alpha p + \beta$$

p = precipitation

$$s_{hh}(\omega) / s_{yy}(\omega) = \frac{\alpha^2}{(s^2\omega^2 + a^2)}$$

$$dZ_h = dZ_y / (isw + a)$$

$$dZ_y^* = dZ_y^*$$

$$s_{yh}(\omega)d\omega = E[dZ_y dZ_y^*] = \frac{E[dZ_y dZ_y^*]}{(is\omega + a)}$$

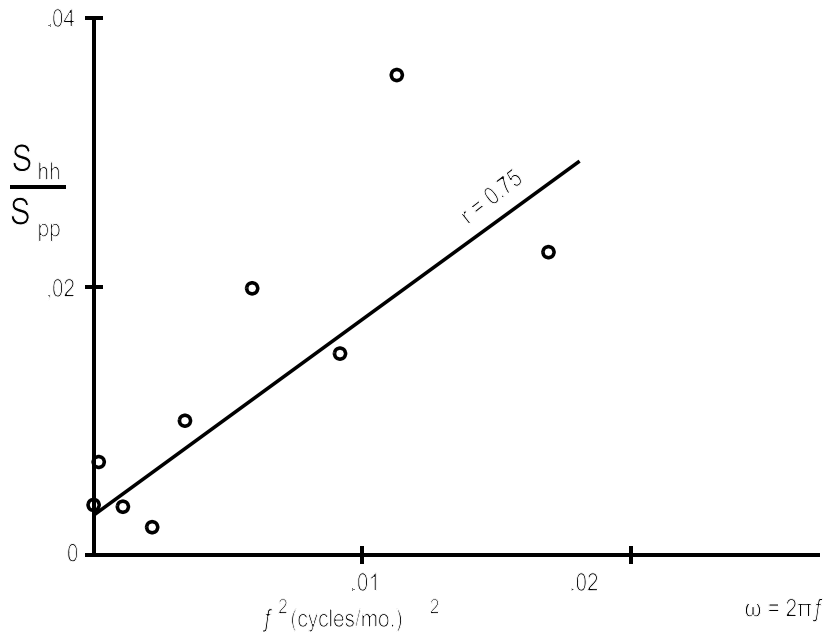
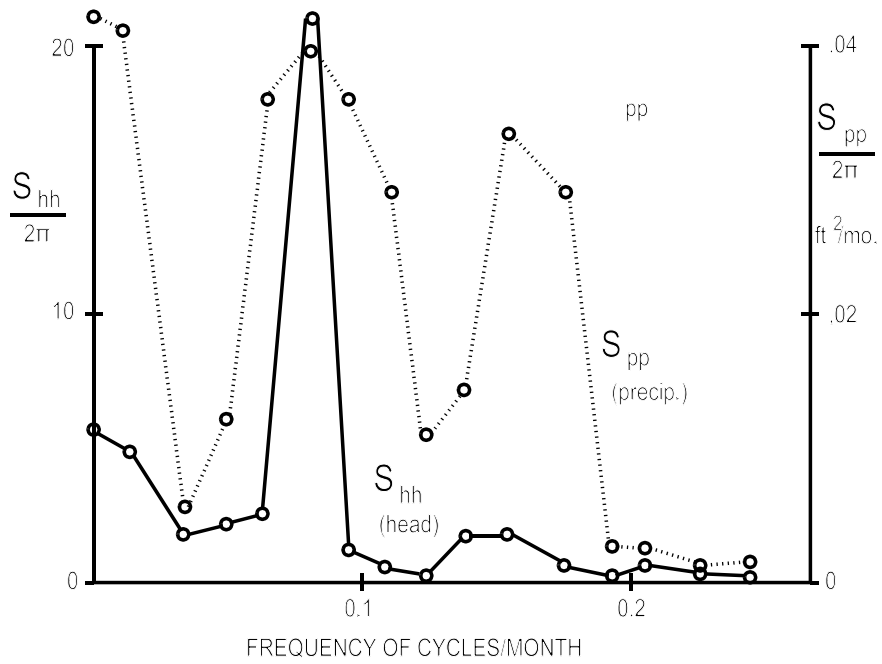
Cross-covariance & Cross-spectrum:

$$R_{yh}(\gamma) = E[y'(t + \gamma)h'(t)] = \int_{-\infty}^{\infty} s_{yh}(\omega)e^{i\omega\gamma} d\omega$$

$$s_{yh}(\omega) = s_{yy} \frac{(a + i\omega s)}{(a^2 + \omega^2 s^2)} = \text{Cov}_{yh}(\omega) - iQ_{yh}(\omega)$$

$$\tan \theta_{yh}(\omega) = \frac{Q_{yh}(\omega)}{\text{Cov}(\omega)}$$

WILMINGTON, MASS. 1960-1972



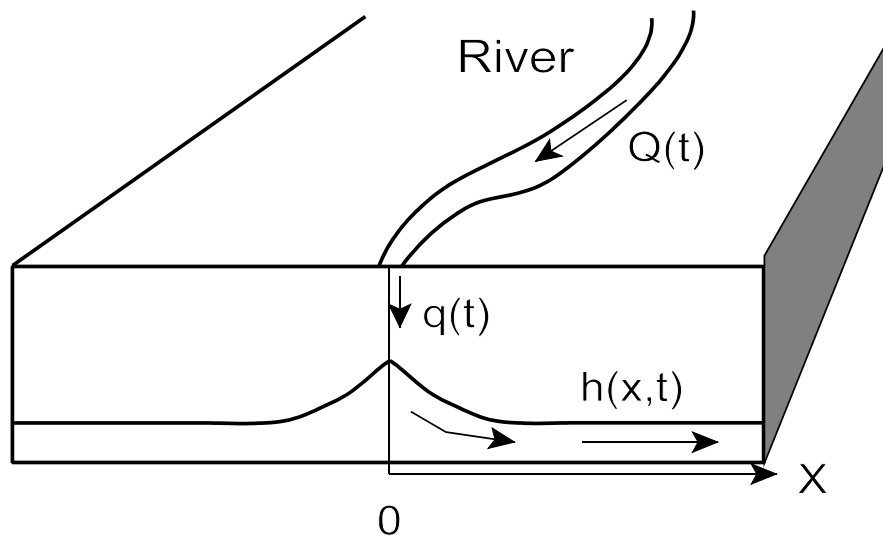
$$\frac{S_{pp}}{S_{hh}} = \frac{s^2}{\alpha^2} \omega^2 + \frac{a^2}{\alpha^2} \Rightarrow s/a = 3.5mo$$

Stochastic Stream-Aquifer Model

Duff et al. 1978, Recharge and Ground Water Conditions in the Western Region of the Roswell Basin, New Mexico. (New Mexico *WRRI*. Rept. No. 100)

- Use
1. Stream flow time series
 2. Water level (Well) time series

to estimate S.T. & a leakage factor



Model: Assume a 2-D depth-average model is adequate to describe the flow in the aquifer.

$$T \frac{\partial^2 h}{\partial x^2} + T \frac{\partial^2 h}{\partial y^2} = S \frac{\partial h}{\partial t} - \varepsilon(x,t) \quad (1)$$

where $\varepsilon(x,t)$ denotes the recharge to the aquifer

$$\varepsilon(x,t) = q(t)\delta(x) \quad (2)$$

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Assume the regional water table gradient

$$\frac{\partial h}{\partial y} \cong \textit{const} \rightarrow \frac{\partial^2 h}{\partial y^2} = 0$$

Then the 2-D governing flow equation is reduced to a 1-D equation, describing flow along the x direction.

$$T \frac{\partial^2 h}{\partial x^2} = S \frac{\partial h}{\partial t} \quad (3)$$

where the boundary condition is given as

$$2T \frac{\partial h}{\partial x} \Big|_{x=0} = -q(t) \quad (4)$$

Only half of the total recharge to the aquifer is assumed to flow in the direction under analysis.

Assume that recharge is a fraction of the river discharge, i.e.,

$$q(t) = aQ(t)$$

where $Q(t)$ is total discharge of the river [L^3/T].

$$q(t) = \text{recharge from the river to the aquifer} \left[\frac{L^2}{T} \right].$$

$$a = \text{constant coefficient (leakage)} \left[\frac{1}{L} \right].$$

Assume that the river discharge $Q(t)$ is a 2nd order stationary stochastic process in time and leakage is a constant. The recharge, $q(t)$, is a stochastic process then.

$$Q(t) = \bar{Q}(t) + Q'(t) \quad \text{and} \quad q(t) = \bar{q}(t) + q'(t) \quad (5)$$

where $\bar{Q}(t) = E[Q(t)]$; $Q'(t)$ is the perturbation and $E[Q'(t)]=0$. Similarly, $\bar{q} = E[q(t)]$ and $q'(t)$ is the perturbation, which is a stochastic process and $E[q'(t)]=0$. Similarly,

$$h(x,t) = \bar{h}(x,t) + h'(x,t) \quad \bar{h}(x,t) = E[h(x,t)] \quad E[h'(x,t)] = 0 \quad (6)$$

Notice that $h(x,t)$ is a stochastic process in t only. Substituting (5) and (6) into (3) and taking the expected value lead to a mean flow equation:

$$T \frac{\partial^2 \bar{h}}{\partial x^2} + \bar{q} \delta(x) = S \frac{\partial \bar{h}}{\partial t} \quad (7)$$

The perturbation equation for (3) becomes

$$S \frac{\partial h'}{\partial t} = T \frac{\partial^2 h'}{\partial x^2} \quad (8)$$

with the boundary condition

$$2T \frac{\partial h'}{\partial x} \Big|_{x=0} = -q' = -aQ'(t) \quad (9)$$

Next, we use the Fourier-Stieltjes integral to represent the perturbations:

$$q'(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_q(\omega)$$

and

$$h'(x, t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_h(x, \omega)$$

Then we have a stochastic partial differential equation

$$S \frac{\partial}{\partial t} \int_{-\infty}^{\infty} e^{i\omega t} dZ_h(x, \omega) = T \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} e^{i\omega t} dZ_h(x, \omega)$$

and

$$\int_{-\infty}^{\infty} e^{i\omega t} S i \omega dZ_h(x, \omega) = \int_{-\infty}^{\infty} e^{i\omega t} T \frac{\partial^2}{\partial x^2} dZ_h(x, \omega) \quad (10)$$

Equation (10) implies that

$$S i \omega dZ_h(x, \omega) = T \frac{\partial^2}{\partial x^2} dZ_h(x, \omega) \quad (11)$$

which is a second-order differential equation. Assume the solution to (11) takes the form

$$dZ_h(x, \omega) = C_1 e^{-\lambda x} \quad (12)$$

where C_1 is unknown constant. Substitute (12) into (11), we have the characteristic equation of (11),

$$T\lambda^2 - i\omega S = 0 \quad (13)$$

Solution to the characteristic equation is

$$\lambda = \pm \sqrt{\frac{i\omega S}{T}} \quad (14)$$

and the solution to the general solution to the stochastic equation becomes

$$dZ_h(x, \omega) = C_1 e^{-\frac{\sqrt{i\omega S}}{T}|x|} \quad (15)$$

We use the boundary condition to determine the unknown constant, C_1

$$2T \frac{\partial h'(x,t)}{\partial x} \Big|_{x=0} = -q'(t) = -aQ'(t) \quad (16)$$

Application of the Fourier-Stieltjes integral to equation (16) leads to

$$\frac{\partial dZ_h(x,\omega)}{\partial x} \Big|_{x=0} = -\frac{adZ_Q(\omega)}{2T} \quad (17)$$

Substituting equation (15) into (17), (1) and letting $x=0$, we have

$$\frac{dZ_h}{dx} = -C_1 \sqrt{\frac{i\omega S}{T}} = -\frac{adZ_Q}{T}$$

and

$$C_1 = \frac{adZ_Q}{2\sqrt{i\omega ST}}$$

Therefore, the particular solution to equation (8) with the boundary condition, equation (9), is

$$dZ_h(x,\omega) = \frac{adZ_Q}{2\sqrt{i\omega ST}} e^{-\sqrt{\frac{i\omega S}{T}}|x|}$$

Recall that $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

Then, spectrum of h becomes

$$S_{hh}(\omega)d\omega = E[dZ_h dZ_h^*] = E[dZ_Q dZ_Q^*] \frac{\exp\left[-\sqrt{\frac{2\omega S}{T}}x\right] a^2}{4\omega ST}$$

or

$$\frac{S_{hh}(\omega)}{S_{QQ}(\omega)} = \left\{ \frac{a^2}{4ST\omega} \exp\left[-\sqrt{\frac{2\omega S}{T}}x\right] \right\} \quad (\text{Transfer Equation})$$

$$\ln \left[\omega \frac{\hat{S}_{hh}}{\hat{S}_{QQ}} \right] = -x \sqrt{\frac{2S}{T}} \sqrt{\omega} + \ln \left[\frac{a^2}{4ST} \right]$$

where $\hat{S}_{hh}(\omega)$ and $\hat{S}_{QQ}(\omega)$ are sample spectra of h and Q. Use the linear regression to determine β_1 and β_2

Procedure:

Step 1. Calculate covariance fns of h and a i.e.

$$R_{hh}(\gamma), R_{QQ}(\gamma)$$

Step 2. Take inverse Fourier Transform of $R_{hh}(\gamma)$ & $R_{QQ}(\gamma)$
use F.F.T. or other algorithms.

$$\hat{S}_{hh}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\gamma} R_{hh}(\gamma) d\gamma$$

$$\hat{S}_{QQ}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\gamma} R_{QQ}(\gamma) d\gamma$$

see {Spectral Analysis and its applications by G.M. Jenkins & D.G. Watts, 1968}

Step 3. Smooth the spectra using filters

Take time series course.

Step 4. Linear regression analysis.

β_2 is used to estimate a if S is known

β_1 is used to estimate S/T ratio.

Cross-Spectrum S_{qh}

$$S_{qh}d\omega = E[dZ_Q dZ_h^*] = E[dZ_Q dZ_Q^*] \frac{\exp\left[-\sqrt{\frac{-i\omega Sx}{T}}\right]}{2\sqrt{-i\omega ST}}$$

$$\text{Let } \alpha = \sqrt{\frac{\omega SX}{T}}, \quad \beta = 2\sqrt{\omega ST}$$

$$S_{qh} = S_{QQ} \frac{\exp[-\alpha(-i)^{1/2}]}{\beta(-i)^{1/2}}$$

$$\sqrt{-i} = e^{-i\pi/4} = \left[\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right]$$

$$S_{qh} = \frac{S_{QQ}}{\beta} e^{\frac{i\pi}{4} - \frac{\sqrt{2}}{2}\alpha + i\frac{\sqrt{2}}{2}\alpha} \quad \text{-----(a)}$$

recall

$$Co_{\theta} - iQ_{\theta} = \frac{S_{QQ}}{\beta} e^{\frac{\sqrt{2}}{2}\alpha} \left[\cos\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\alpha\right) + i \sin\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\alpha\right) \right]$$

$$S_{qh} = Ae^{i\theta_{qh}} \quad \text{-----(b)}$$

where

$$A(\omega) = \sqrt{(Co)^2 + (Q_h)^2} \quad \text{Amplitude Spectrum}$$

$$Q_{Qh}(\omega) = \tan^{-1} \left[\frac{-Q_{Qh}}{Co_{Qh}} \right] \quad \text{Phase Spectrum}$$

From (a) & (b)

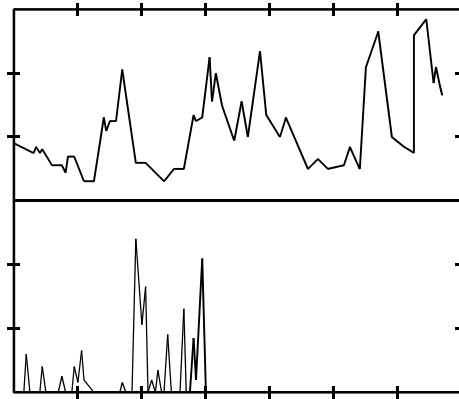
$$Q_{Qh} = \frac{\pi}{4} + \sqrt{\frac{\omega SX}{2T}}$$

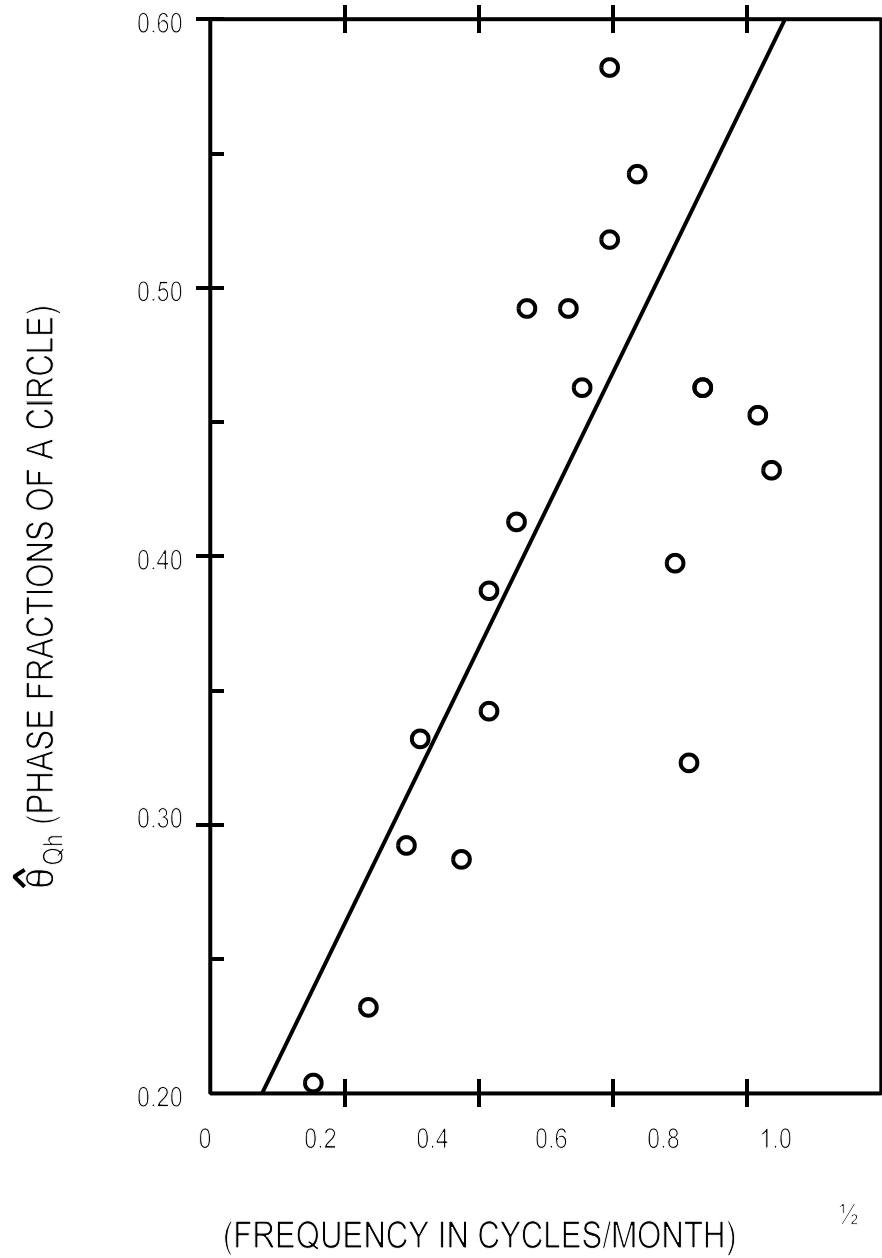
$$Q_{Qh} = \frac{\pi}{4} + \sqrt{\frac{SX}{2T}} \sqrt{\omega}$$

PAGE OF FIGURES -

(SEE EXAMPLE OF WHAT IT WOULD LOOK LIKE IF I DID THEM)

(IF YOU LIKE I CAN DO THE OTHERS LIKE THIS)





should have fix $b_0 = ?$
 $= 56.5^\circ \approx 45^\circ$ $b_1 = 0.52 = 3.27$ radians $R=0$

Figure 13: The phase in fractions of a circle plotted against square root of frequency.

R = correlation coefficient. Data in Table IV.