VARIOUS FORMS OF DARCY'S LAW FOR FIELD-SCALE PROBLEMS(535-2d.wp)

1 - D

$$q = -K\frac{dh}{dx}$$

where the specific discharge, q, conductivity, K, and gradient, dh/dx are scalar.

2 - D

Isotropic Media	Anisotropic Media
$\underline{\underline{q_x}} = -K\frac{\partial h}{\partial x}$	$q_x = -K_x\frac{\partial h}{\partial x}$
$\underline{\underline{q_y}} = -K\frac{\partial h}{\partial y}$	$q_y = -K_y\frac{\partial h}{\partial y}$

where $q_x, q_y, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}$ are vectors characterized by magnitudes and directions.

Quotient Rule: If the result of a mathematic quantity multiplying a vector is a vector, the mathematic quantity must be either a scalar or a tensor. For example,

$$\text{If we let} \quad \vec{J} = \left[-\frac{\partial h}{\partial x},\ -\frac{\partial h}{\partial y}\right]$$

$$\vec{q} = [\quad] \cdot \vec{J}$$

Therefore, [ ] has to be a scalar or a tensor. The hydraulic conductivity of an isotropic medium is a scalar, and that of an anisotropic medium is a tensor.

§ Darcy's Law in 2-D or 3-D

Vector notation:

$$\vec{q} = -\overline{\overline{K}} \cdot \nabla h$$

where the double over bars denote a tensor. One common notation used is the expression based on Einstein's summation convention:

$$q_i = K_{ij}J_j$$

where i and j = 1, 2 or 3 or x, y, z, and J_1, J_2, J_3 = $-\partial h/\partial x$, $-\partial h/\partial y$, and $-\partial h/\partial z$, respectively. Einstein's summation convention says that in any product terms a suffix (subscript or superscript) repeated twice (and only twice) is held to be summed over its range of values. For instance, Darcy's Law in three-dimensions is:

$$q_1 = K_{11}J_1 + K_{12}J_2 + K_{13}J_3$$
$$q_2 = K_{21}J_1 + K_{22}J_2 + K_{23}J_3$$
$$q_3 = K_{31}J_1 + K_{32}J_2 + K_{33}J_3$$

or in 2-D

$$q_1 = K_{11}J_1 + K_{12}J_2$$
$$q_2 = K_{21}J_1 + K_{22}J_2$$

§ Matrix Forms

3-D

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}$$

2-D

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

where the matrices, K_{ij} or $[K]$ define the hydraulic conductivity tensor of the porous medium in 3-D and 2-D flow situations, respectively. Note that a tensor is a quantity in general (matrix) that satisfies a specific coordinate transformation rule.

§ Properties of Hydraulic Conductivity Tensor

1) a symmetric tensor of rank 2
   rank 2 $\Rightarrow$ two indices, i, j
   symmetry $\Rightarrow$ $K_{ij} = K_{ji}$ where $i \neq j$

$$K_{12} = K_{21}\ ,\ K_{13} = K_{31}\ ,\ K_{32} = K_{23}$$

3-D Case:

$$\underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}}_{9\ components} \overset{reduce}{\Rightarrow} \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ & K_{22} & K_{23} \\ & & K_{33} \end{bmatrix}}_{6\ components}$$

2-D Case:

$$\underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}}_{4\ components} \overset{reduce}{\Rightarrow} \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ & K_{22} \end{bmatrix}}_{3\ components}$$

2) when $K_{ij} = K$ (a const.) for $i = j$ & $K_{ij} = 0$, for $i \neq j$

The medium is called <u>isotropic</u>, i.e., the hydraulic conductivity value at a point does not vary with direction.

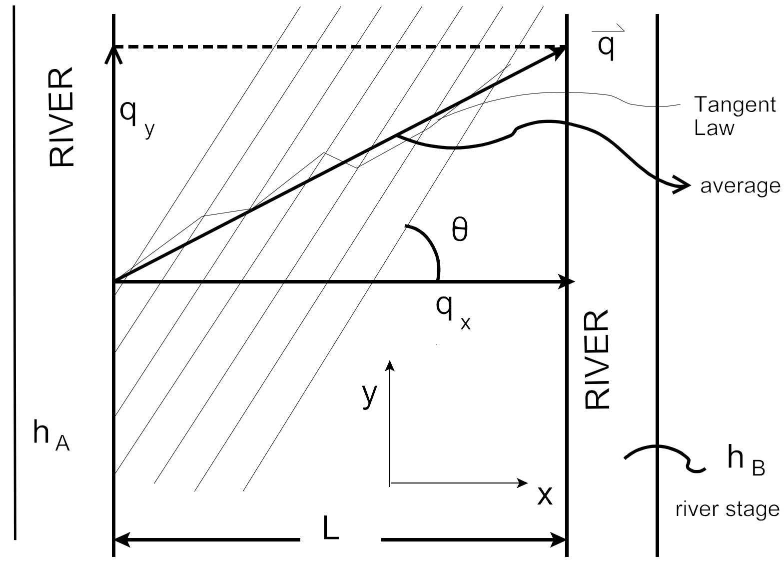
$$\begin{bmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{bmatrix} \Rightarrow \underset{\substack{\downarrow \\ Kronecker\ delta}}{K\delta_{ij}} \Rightarrow \underset{\substack{\uparrow \\ scalar}}{K} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) when $K_{ij} \neq$ const. for $i = j$, the porous medium is called **anisotropic**, i.e., the medium has a higher hydraulic conductivity in one direction than in other directions.

$K_{ij} \neq 0$, for $i \neq j$ (unless in the principal directions)

i.e., K_{ij} should have 9 components in 3-D, 4 in 2-D

§ Physical meaning of the mixed components K_{12}, K_{21}, K_{13}...



Darcy's Law in 2-D

$$q_x = K_{xx}J_x + K_{xy}\cancelto{0}{J_y}$$

$$q_y = K_{yx}J_x + K_{yy}\cancelto{0}{J_y}$$

$$q_x = K_{xx}J_x$$

$$q_y = K_{yx}J_x$$

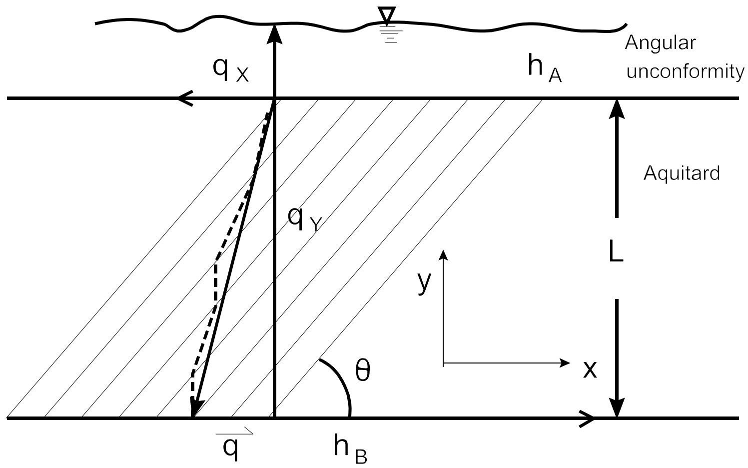
$$J_x = \frac{-\left(h_B - h_A\right)}{L} \neq 0$$

$$J_y = 0$$

$K_{yx} = \dfrac{q_x}{J_x}$, $K_{yx} = \dfrac{q_y}{J_x} \Rightarrow$ the contribution of J_x to the specific discharge $\vec{q}$ in the y-direction.

$$\theta = 0\,, \quad K_{xy}\,, K_{yx} = 0$$

$$\theta = 90^\circ\,, \quad K_{xy}\,, K_{yx} = 0$$



Darcy's Law

$$q_x = \cancelto{0}{K_{xx}J_x} + K_{xy}J_y$$
$$q_y = \cancelto{0}{K_{yx}J_x} + K_{yy}J_y$$

$$J_x = 0$$
$$J_y = -\frac{(h_B - h_A)}{L} \neq 0$$

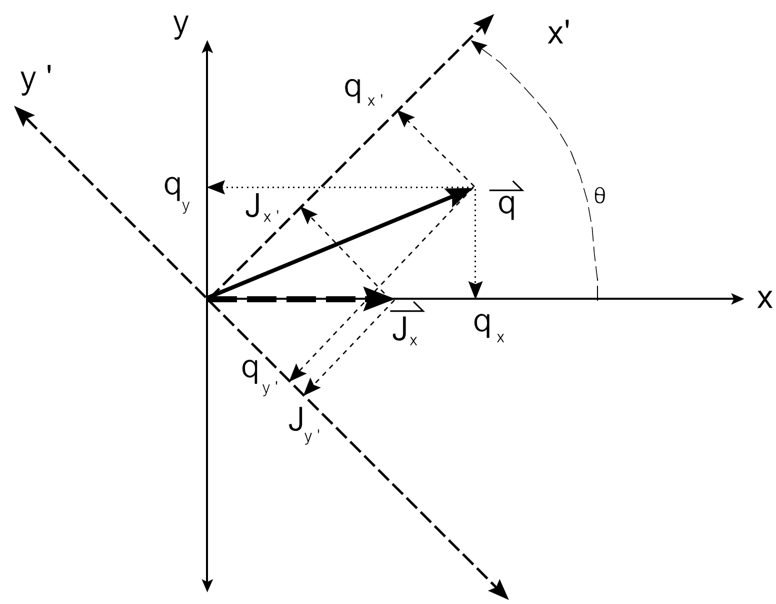
$$q_x = K_{xy}J_y \qquad q_x / J_y = K_{xy}$$
$$q_y = K_{yy}J_y \qquad q_y / J_y = K_{yy}$$

Notice that

$$\theta = 0\,, \quad K_{xy} = K_{yx} = 0$$
$$\theta = 90^\circ\,, \quad K_{xy} = K_{yx} = 0$$

As a result, mixed components depend on the coordinate systems, and they are artifacts.

Alternatively, one can choose a different coordinate system such as the following:



where
$$q_x = K_{xx} J_x \qquad q_{x'} = K_{x'x'} J_{x'}$$
$$q_y = K_{yx} J_x \quad \text{or} \quad \underbrace{q_{y'} = K_{y'y'} J_{y'}}_{no\ mixed\ components!}$$

$K_{yx} \rightarrow$ is an artifact.

In this coordinate system, there are no mixed components in the hydraulic conductivity tensor. However, this coordinate system must be in the principal directions.

4) The principal directions of the hydraulic conductivity of the anisotropic medium are the directions for which $K_{ij} = 0$ for all $i \neq j$ and $K_{ij} \neq 0$ for $i = j$.

$$[K] = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \qquad [K] = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$

In general, the principal directions of an anisotropic medium composed of alternating layers are the directions parallel to the bedding planes and those perpendicular to the bedding planes.

5) The transformations from any x, y coordinates to the principal x', y' coordinates can be obtained by using the following relationships **(2-D case)**

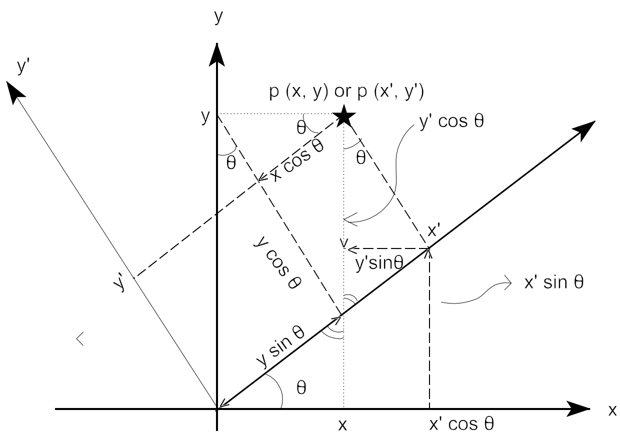
$$\begin{matrix} K_{x'x'} \\ K_{y'y'} \end{matrix} = \frac{K_{xx} + K_{yy}}{2} \pm \left[\left(\frac{K_{xx} - K_{yy}}{2} \right)^2 + K_{xy}{}^2 \right]^{1/2}$$

or

$$\begin{matrix} K_{xx} \\ K_{yy} \end{matrix} = \frac{K_{x'x'} + K_{y'y'}}{2} \pm \frac{K_{x'x'} - K_{y'y'}}{2} \cos 2\theta$$

$$K_{xy} = \frac{K_{x'x'} - K_{y'y'}}{2} \sin 2\theta$$

2-D Coordinate Transformation



From x', y' to x, y :

$$\begin{pmatrix} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{pmatrix}$$

From x, y to x', y' :

$$\begin{pmatrix} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{pmatrix}$$

θ is a positive value

$\cos\theta$, $\sin\theta \rightarrow$ directional cosines

***K*-Tensor Rotation**:

The tensor transformation is given by $\left[K\right] = \left[D\right]\left[K'\right]\left[D\right]^T$ where [*K*] is the conductivity tensor in the *x*-*y* coordinate system, [*D*] is the directional cosine matrix, $\left[K'\right]$ is the conductivity tensor in the $x' - y'$ coordinate system, and $[D]^T$ is the transpose of [*D*]. More specifically, a two-dimensional tensor for example can be transformed by:

$$\begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} = \overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{D} \overbrace{\begin{bmatrix} K_{x'x'} & 0 \\ 0 & K_{y'y'} \end{bmatrix}}^{\substack{Principal \\ Ks}} \overbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}^{D^T}$$

$$= \begin{bmatrix} K_{x'x'}\cos\theta & -K_{y'y'}\sin\theta \\ K_{x'x'}\sin\theta & K_{y'y'}\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} K_{x'x'}\cos^2\theta + K_{y'y'}\sin^2\theta & K_{x'x'}\cos\theta\sin\theta - K_{y'y'}\sin\theta\cos\theta \\ K_{x'x'}\sin\theta\cos\theta - K_{y'y'}\cos\theta\sin\theta & K_{x'x'}\sin^2\theta + K_{y'y'}\cos^2\theta \end{bmatrix}$$

Recall the following relationship:

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1) \qquad \sin^2\theta = \frac{1}{2}(-\cos 2\theta + 1)$$

$$K_{xx} = K_{x'x'}\cos^2\theta + K_{y'y'}\sin^2\theta$$

$$= \frac{K_{x'x'}}{2}\cos 2\theta + \frac{K_{x'x'}}{2} + \frac{-K_{y'y'}}{2}\cos 2\theta + \frac{K_{y'y'}}{2}$$

$$= \frac{K_{x'x'} + K_{y'y'}}{2} + \frac{(K_{x'x'} - K_{y'y'})}{2}\cos 2\theta$$

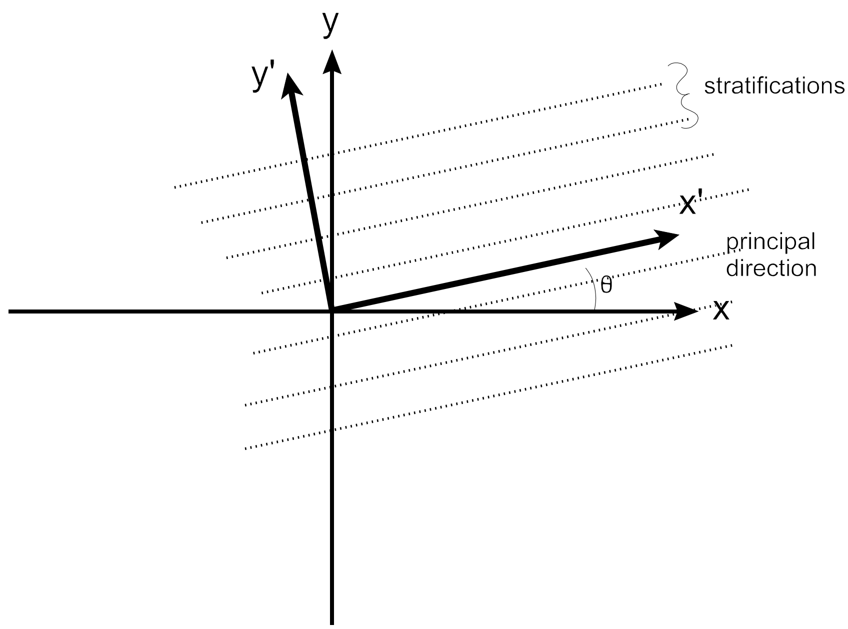
$$K_{yx} = K_{x'x'}\sin\theta\cos\theta - K_{y'y'}\cos\theta\sin\theta \qquad \text{since } \sin 2\theta = 2\sin\theta\cos\theta$$

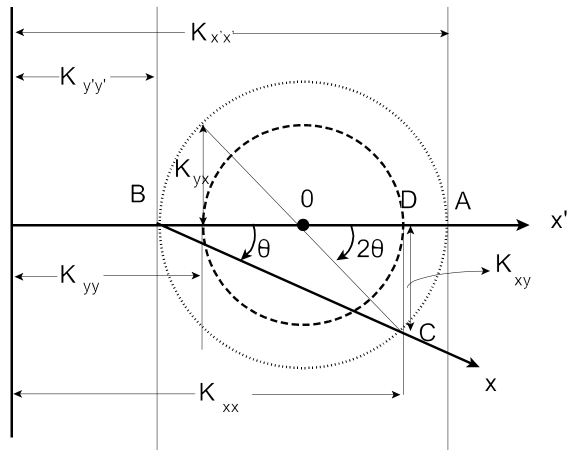
$$K_{yx} = \frac{K_{x'x'} - K_{y'y'}}{2}\sin 2\theta = K_{xy}$$

Note K_{xy} and K_{yx} can be negative, depending on the value of θ. This is another proof showing that the mixed term is a mathematical artifact.

§ Mohr's Circle Representation of a Hydraulic Conductivity Tensor in 2-D.

A most convenient way of representing the above conductivity tensor transformation in a graphic form is by using Mohr's circle. For example,





Based on this figure, it is easy to see the following transformation rules if the principal *K*s are known.

$$\text{Center of the circle } = \frac{K_{x'x'} + K_{y'y'}}{2}$$

$$\text{Radius of the circle } = \frac{K_{x'x'} - K_{y'y'}}{2}$$

$$K_{xx} = \frac{K_{x'x'} + K_{y'y'}}{2} + \frac{K_{x'x'} - K_{y'y'}}{2}\cos 2\theta$$

$$K_{yy} = \frac{K_{x'x'} + K_{y'y'}}{2} - \frac{K_{x'x'} - K_{y'y'}}{2}\cos 2\theta$$

$$K_{xy} = -\frac{K_{x'x'} - K_{y'y'}}{2}\sin 2\theta \qquad \theta \text{ is negative.}$$

Constraint on θ

$$\frac{K_{x'x'}}{K_{y'y'}} = \frac{K_{xx} - K_{yy}\sin^2\theta}{K_{yy} - K_{xx}\sin^2\theta} > 1 \qquad \text{if } K_{x'x'} > K_{y'y'}$$

Similarly, the transformation of the conductivity tensor from the *x*-*y* coordinate system to the principal direction coordinate system is given by:

$$\text{Center of the small circle } = \frac{K_{xx} + K_{yy}}{2}$$

$$\text{Radius of the small circle } \; OD = \frac{K_{xx} - K_{yy}}{2}$$

$$\overline{OC}^2 = \overline{OD}^2 + \overline{CD}^2$$

$$= \left(\frac{K_{xx} - K_{yy}}{2}\right)^2 + K_{xy}^2$$

$$\overline{OC} = \left[\left(\frac{K_{xx} - K_{yy}}{2}\right)^2 + K_{xy}^2\right]^{1/2}$$

$$K_{x'x'} = \frac{K_{xx} + K_{yy}}{2} + \left[\left(\frac{K_{xx} - K_{yy}}{2}\right)^2 + K_{xy}^2\right]^{1/2}$$

$$K_{y'y'} = \frac{K_{xx} + K_{yy}}{2} - \left[\left(\frac{K_{xx} - K_{yy}}{2}\right)^2 + K_{xy}^2\right]^{1/2}$$

Property of Hydraulic Conductivity Tensor: In spite of transformation, the hydraulic conductivity tensor must satisfy the following property. That is,

$$K_{x'x'} + K_{y'y'} = K_{xx} + K_{yy}$$

independent of rotation.

Question:

1) Why is the hydraulic conductivity tensor symmetrical? How is it related to the concept of REV?

Discussions

Despite treating the field-scale geological medium as a homogeneous or heterogeneous medium, the classical analysis faces many difficulties. First, the validity of Darcy's Law for the equivalent homogeneous field problem has not yet been proven. Since input sources are small compared with the size of the field-scale geological medium, water and contaminant plumes must travel a great distance to encounter all heterogeneities at different scales before an FSREV can be defined. This distance may be greater than the thickness of the field-scale geological medium. As a result, the FSREV may not exist. Second, even if a large FSREV for a given vadose zone exists, no conclusive means is available to obtain effective properties of the equivalent homogeneous field problem, using data from large-scale hydraulic tests. Relying on small-scale hydraulic tests, one would obtain different hydraulic parameter values at various parts of the field-scale geological medium. Then, a theoretically rigorous means to average these different values is needed. Most important of all, we often do not know what we predict, using the effective property. Are the predicted behaviors unbiased estimates of the processes in the field-scale geological medium? Do they bear any statistical significance?

Additionally, sample sizes of our monitoring devices (e.g., sample volumes of a well, tensiometer, lysimeter, time domain reflectrometer probe "TDR", or neutron probe) are generally much smaller than the FSREV. Predictions based upon an assumption of homogeneity are expected to deviate from our point observations unless our observation focuses on the integrated or averaged behavior. Then, a measure of the difference between the prediction and our point observations is imperative. This measure is important for economic and environmental reasons (for instance, selection the location of a pump and treatment well system or determination of the extent of contamination).

The aforementioned issues are critical to the accuracy of our predictions but methodologies to address these issues do not exist in the classical homogeneous approach. Using only a limited amount of data and a general geohydrological description of the field-scale geological medium, the heterogeneous approach also confronts the same issues though it does not invoke the FSREV assumption for the entire vadose zone or the aquifer.

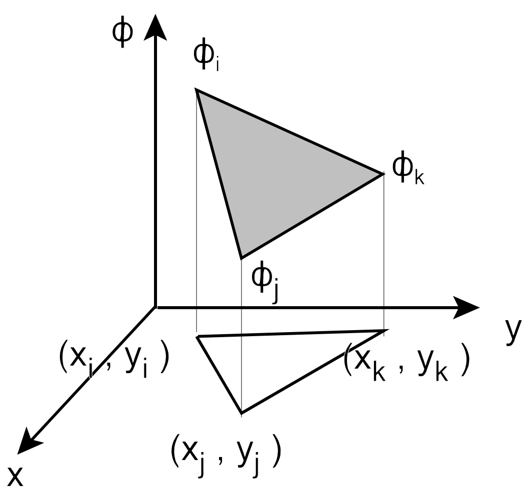
Determination of Hydraulic Gradients

From previous discussions, we know that the hydraulic gradient is a vector that is characterized by magnitude and direction. In a two-dimensional flow field, the hydraulic gradient will consist of components in the *x* and *y* directions. On the other hand, the gradient can be decomposed into three components in three-dimensional flow scenarios. For many practical problems, you may be required to estimate the hydraulic gradient in a two-dimensional aquifer. This can be done easily using a graphic method you learned in field geology for determining the dip and strike of a bed. Here, we will introduce a finite element approach to solve the problem.

Obviously, you must have water level measurements at wells at three different locations to determine the gradient. Suppose we have three water level measurements as illustrated in the figure below. If we assume that the water level varies linearly in space, there must be a plane covering the three water levels. The water level at these three locations are denoted by ϕ_i, ϕ_j, and ϕ_k, whereas the coordinates of the associated locations are (x_i, y_i), (x_j, y_j), and (x_k, y_k). From your vector calculus, the plane is

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y$$

with the condition that α_1, α_2, and α_3 are unknowns. The hydraulic heads at the three locations must be equal to the observations. Substitution of these conditions into the equation for the plane produces the system of equations:



$$\begin{aligned} \phi &= \phi_i \quad at \quad x = x_i \, , \, y = y_i \\ \phi &= \phi_j \quad at \quad x = x_j \, , \, y = y_j \\ \phi &= \phi_k \quad at \quad x = x_k \, , \, y = y_k \end{aligned}$$

Solution to the system of equations yields

$$\alpha_1 = \frac{1}{2A}\left[\left(x_j y_k - x_k y_j\right)\phi_i + \left(x_k y_i - x_i y_k\right)\phi_j + \left(x_i y_j - x_j y_i\right)\phi_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[\left(y_j - y_k\right)\phi_i + \left(y_k - y_i\right)\phi_j + \left(y_i - y_j\right)\phi_k\right]$$

$$\alpha_3 = \frac{1}{2A}\left[\left(x_k - x_j\right)\phi_i + \left(x_i - x_k\right)\phi_j + \left(x_j - x_i\right)\phi_k\right]$$

where $2A$ is the determinant

$$\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2A$$

where A is the area of the triangle.

Substitute for α_1, α_2, and α_3 in the equation for the plane and rearrange results in an equation for ϕ in terms of three shape functions and ϕ_i, ϕ_j, and ϕ_k. That is,

$$\phi = N_i\phi_i + N_j\phi_j + N_k\phi_k$$

where the shape functions are given as

$$N_i = \frac{1}{2A}\left[a_i + b_i x + c_i y\right]$$

$$N_j = \frac{1}{2A}\left[a_j + b_j x + c_j y\right]$$

$$N_k = \frac{1}{2A}\left[a_k + b_k x + c_k y\right]$$

Notice that these shape functions are functions of x and y, and

$$a_i = x_j y_k - x_k y_j\ , \quad b_i = y_j - y_k\ , \quad c_i = x_k - x_j$$
$$a_j = x_k y_i - x_i y_k\ , \quad b_j = y_k - y_i\ , \quad c_j = x_i - x_k$$
$$a_k = x_i y_j - x_j y_i\ , \quad b_k = y_i - y_j\ , \quad c_k = x_j - x_i$$

Therefore, the head value at any point within the plane can be related to any one of the three head values by a set of shape functions that are linear in x and y. This means that the gradients are constant within the plane which can be evaluated as

$$\frac{\partial \phi}{\partial x} = \frac{\partial N_i}{\partial x}\phi_i + \frac{\partial N_j}{\partial x}\phi_j + \frac{\partial N_k}{\partial x}\phi_k$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial N_i}{\partial y}\phi_i + \frac{\partial N_j}{\partial y}\phi_j + \frac{\partial N_k}{\partial y}\phi_k$$

where

$$\frac{\partial N_\eta}{\partial x} = \frac{b_\eta}{2A} \qquad \eta = i, j, k$$

$$\frac{\partial N_\eta}{\partial y} = \frac{c_\eta}{2A} \qquad \eta = i, j, k$$

Therefore,

$$\frac{\partial \phi}{\partial x} = \frac{1}{2A}\left[b_i\phi_i + b_j\phi_j + b_k\phi_k\right]$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{2A}\left[c_i\phi_i + c_j\phi_j + c_k\phi_k\right]$$