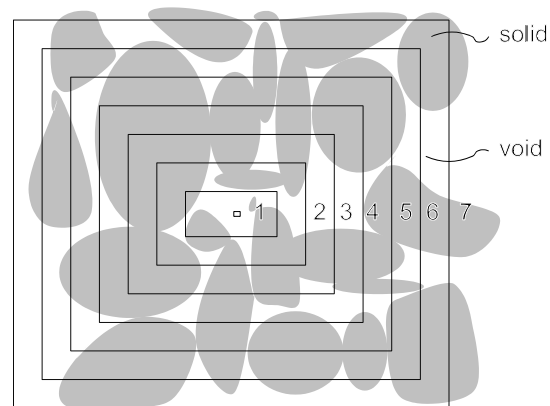


§ Control Volume, Darcian Continua and Representative Elementary Volume

To study water flow and solute transport in porous media, hydrologists have relied on a continuum approach. At pore scale, flow in porous media takes place through a complex network of interconnected pores or openings. To describe such an intricate network in any exact mathematical manner is practically impossible. The basic equation governing fluid flow at the pore-scale level (i.e., the Navier-Stokes equation) thus must be abandoned. Consequently, Darcy's law only considers the average flow behavior over a certain volume of porous media that must be greater than several pores. This volume over which the flow is averaged is then defined as a **control volume (CV)**. Using this CV approach, Darcy's law essentially bypass both the microscopic level, at which we consider what happens to each fluid molecule, and the pore level, at which we consider the flow pattern within each pore and between pores. Our observations of flow in porous media then move to the macroscopic level at which only average phenomena over the control volume are considered. That is to say, Darcy's law is a macroscopic law. The property defined at a point in Darcy's Law thus represents an average property over a CV and the property at every point in space varies smoothly such that our differential calculus applies. The medium and flow are subsequently considered as the **Darcian continua**. This continuum concept is parallel to the continuum hypothesis in fluid mechanics and other branches of sciences.

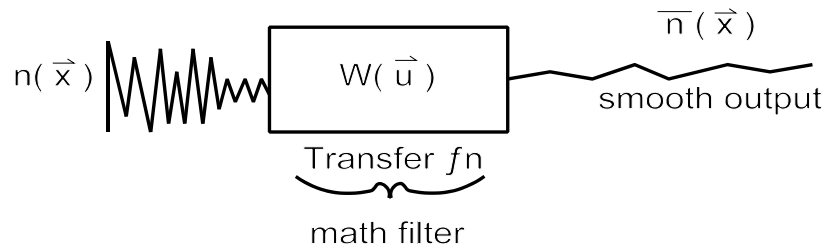
To illustrate the concept of the continuum, let us use the measurement of porosity of a uniformly packed sand column as an example.



Suppose we selected a point as our CV (area #1 in the figure) to define the porosity at a given location of the soil column. Porosity values along the soil column will likely have a value of either one or zero along the soil column, depending on where the point lies. As a result, the porosity value behaves in a discrete fashion in the column and we cannot apply a continuous mathematical function to describe its variation. As CV increases, the value of the porosity tends to fluctuate between one and zero with decreasing amplitudes; its spatial variation becomes smooth in space; continuous mathematical functions apply. Finally, the fluctuation becomes small and the porosity value reaches a stable value. In other words, the porosity value defined by this

CV is almost constant over the entire sand column regardless of the position of the CV. Such a CV that produces a constant porosity value along the column is then defined as a representative elementary volume (R.E.V.), Bear, 1972. This definition of REV has often misled CV as an REV. A proper definition of REV is given below.

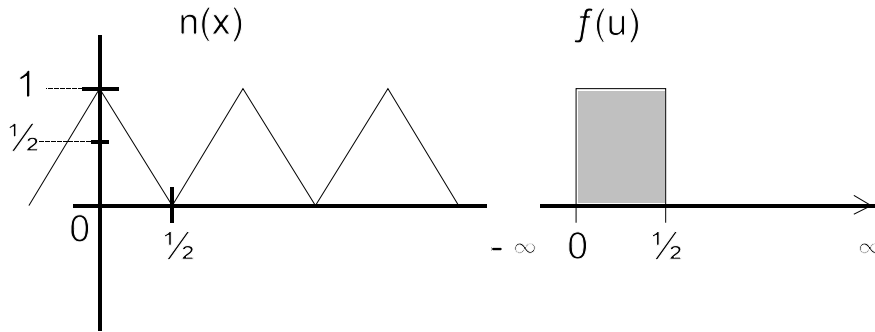
Note that mathematically, the CV concept is equivalent to a convolution integral in mathematics or a spatial averaging concept. A convolution integral can be viewed as a mathematical filter that smooths out the erratic behavior of an input function.



For the porosity example, the convolution integral can be defined as:

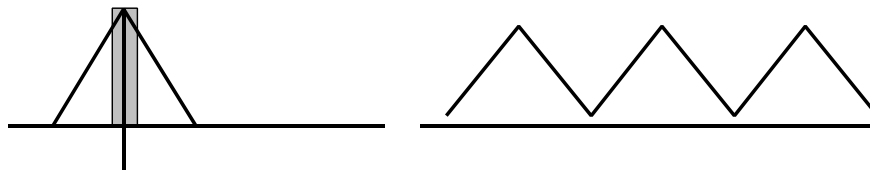
$$N(x) = \int_{-\infty}^{+\infty} f(u)n(x-u)du \quad f(u) \begin{cases} \frac{1}{V} & 0 < u < \frac{1}{2} \\ 0 & otherwise \end{cases}$$

where n is the point measurement of the porosity, $f(u)$ is the averaging process we defined over a CV, and V is the volume of the CV. An illustration of the mathematical process of the convolution integral is given as follows. Consider we are pulling a triangular wave, $n(x)$, slowly through a rectangular window, $f(u)$ which has a width of $1/2$. We then determine the area of the triangular wave appearing in the window as the wave is propagating through.

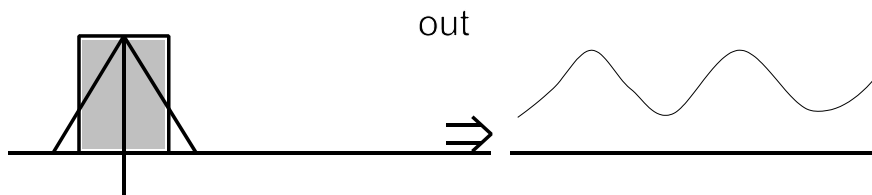


$$N(x) = \int_0^{\frac{1}{2}} 1 \cdot n(x-u) du + \int_0^{\frac{1}{2}} 0 \cdot n(x-u) du$$

The above integral represents the area under $n(x)$ and confined by $f(u)$. Due to the large width of the window, the area we calculate, $N(x)$, will be much smoother than the original triangular wave. However, if the width of the window is very narrow, $N(x)$ is likely to have the same kind of shape of the triangular wave. As the size of the window increases, $N(x)$ becomes much smoother and eventually becomes a constant over the entire distance.



If we apply the CV approach to a porous medium at the laboratory scale, we may classify



the medium as heterogeneous or homogeneous, depending on the size of the CV. If the size of the CV is small, the property is likely to vary spatially. The medium is called **heterogeneous** although the spatial variation of the property is smooth. If progressively larger CV's are used, we find the value of the property tends toward a constant value everywhere in the medium. These volumes are defined as REV's (**Representative Elementary Volumes**) and the medium is classified as a **homogeneous** medium. Specifically, the property defined over a REV of a medium must represent the property of the medium regardless of locations (translational invariant). To do so, the size of the REV must be large enough that it includes all representative heterogeneities in the medium. This REV concept is analogous to the minimum sample volume required in statistical analysis. This volume must be large enough so that the statistics of samples in the volume represents a given population. The REV also has to be small so that spatially continuous functions can be used to describe properties of the medium. As an example, the volume of a REV for a homogeneous sand box must be several times of the average grain diameter of the sand but smaller than the sand box, itself.

The size of the REV is scale-dependent. That is, one may define a REV for a laboratory-scale problem but this REV may not be applicable to field-scale problems. Furthermore, the existence of a REV for some medium properties will be time-dependent. For medium properties such as porosity and bulk density that are not associated with flow, existence of the REV is independent of time. On the other hand, the REV for properties related to flow process (such as hydraulic conductivity) can only be defined when the REV requirement with respect to flow is met. That is to say, the flow process must take sufficient time to experience many heterogeneities in the medium. For example, the wetting front from a point source in a sand box must evolve to a certain size before the REV concept can be applied. This time-dependent phenomenon is nonetheless obscured by the fact that in a homogeneous soil column, the size of heterogeneity is on the order of grain diameters and the time for flow to reach the REV is insignificant compared with the time scale of our experiments. Thus, the time-dependent phenomenon is neglected.

Note that one often uses the entire soil column as a CV and defines some hydraulic properties. These properties are basically properties averaged over the entire volume and they do not necessarily satisfy the concept of REV.

Scale of measurement (or interest, objective)

The pressure head in Darcy's Law represents the average pressure head over the REV, which is not necessarily the same as the pressure head measured in a single pore. To be consistent with the Darcian continuum concept, our measurement scale for head and concentration must be at least as large as the size of the REV. Generally, the propagation of pressure waves in porous media is a highly diffusive process so that the difference between the pressure head within a pore and the average is negligible and thus this requirement not critical. On the other hand, the propagation of moisture in unsaturated porous media is generally not as diffusive as the pressure wave, especially at low degrees of saturation. Nevertheless, this requirement may not be critical during unsaturated flow experiments in soil columns. First, the size of the water source and the column length in the experiment are usually much larger than that of the heterogeneity (i.e., variations in pore size and geometry). Therefore, after the moisture plume is displaced over a distance encompassing several

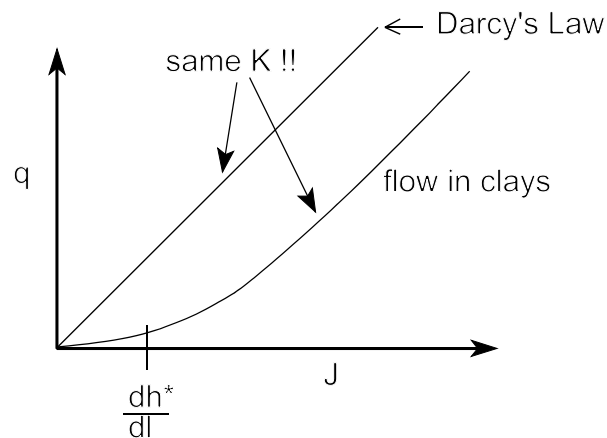
pores, its behavior is well described by Darcy's law, reflecting the existence of the REV. The cross-sectional area at the end of the column, where the integrated moisture outflow is collected, is thus equal to or greater than the REV. However, inconsistencies between the measurement scale and REV may occur. For instance, a flush mounted tensiometer in a soil column may not sample a volume that is equivalent to the REV embedded in Darcy's law.

The Validity of Darcy's Law

Darcy's law is not universal and it works only for some groundwater flow regimes. It has lower and upper limits:

1) The Lower Limit

Darcy's Law fails to hold at very low values of head gradient in some fine materials such as compacted clays.



According to Darcy's law, any hydraulic head gradient, no matter how small, will cause the motion of water. On the contrary, many experiments have shown that the hydraulic head gradient must exceed some threshold value $\frac{dh^*}{dl}$ in order to cause water movement in clayey soils. The failure of Darcy's law in this case can be attributed to the fact that important electrostatic forces exist, resulting from electrical charge imbalances within the mineral structure. At low hydraulic head gradients, these forces cannot be neglected in the formation of Poiseuille's Equation. Nonetheless, this problem is not significant in field-scale aquifers where we have many other difficulties to resolve. At any rate, flow regimes that do not satisfy Darcy's Law (i.e., linear relationship between q and gradient) are called **Non-Darcian flow**.

Before we discuss the upper limit of Darcy's law, let's we introduce Reynold's number for groundwater flow.

§ Reynold's Number for Groundwater Flow.

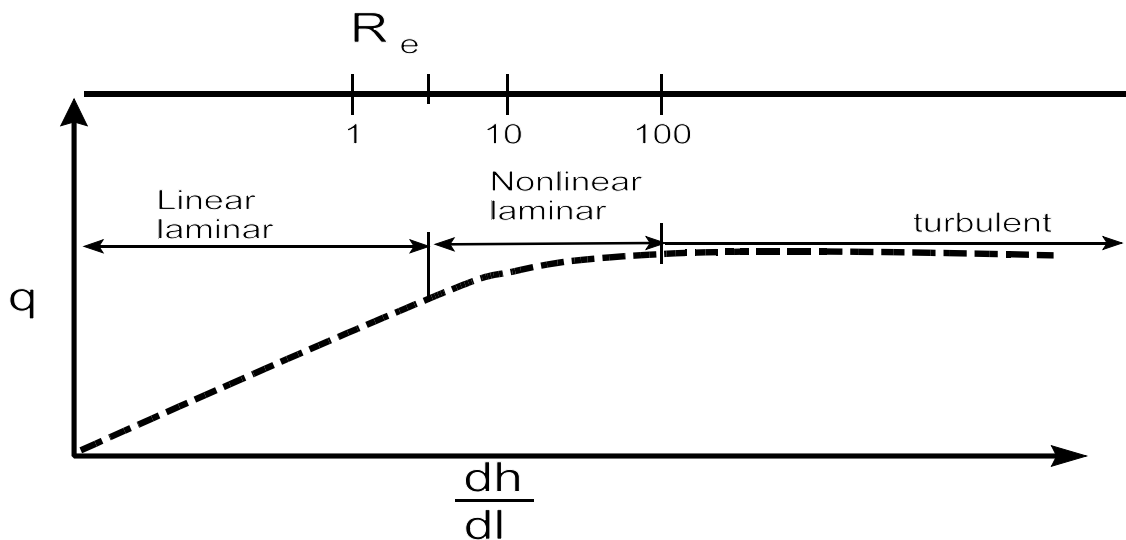
$$R_e = \frac{q\rho d}{\mu} = \frac{qd}{\nu} \rightarrow \frac{\textit{inertial force}}{\textit{viscous force}}$$

where

q	=	specific discharge,	$\nu = \frac{q}{n}$
d	=	a characteristic length	
	=	d_{10}	
μ	=	dynamic viscosity	
$\nu = \frac{\mu}{\rho}$	=	kinematic viscosity	

2) Upper Limit

According to Darcy's Law, the specific discharge q should be linearly proportional to the head gradient. Flow behaviors deviating from the linear relation (i.e., Darcian behavior) however have been observed at large dh/dl values or at $10 \leq R_e < 100$. Under these conditions, the relation between q and dh/dl is no longer found to be linear although the flow is still under laminar flow



regimes. At $R_e \gg 100$, turbulent flow with streamlines crossing each other takes place. Under the turbulent flow regime, q is no longer correlated with dh/dl and flow exhibits chaotic behavior

(random motion). It becomes difficult to describe this flow behavior quantitatively.

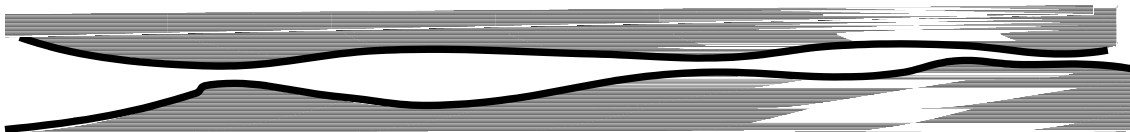
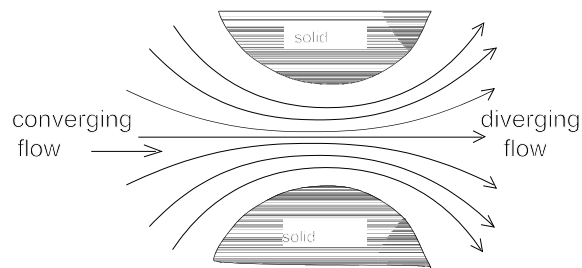
When $10 < R_e < 100$, nonlinear laminar flow regime exists. A modified Darcy's law has been developed.

$$\text{large } \frac{dh}{dl} \rightarrow \text{large } q \rightarrow \text{large } R_e$$

\rightarrow *inertial force* \gg *viscous force*

\rightarrow *fluid acceleration* \rightarrow *important*

Where does the acceleration come from?



Because Darcy's law is based on steady flow analysis, the local acceleration term is zero but the convective acceleration becomes important. That is to say, changes in velocity due variations in pore radius is significant at large flow rates. Consideration of the importance of convective acceleration leads to Forcheimer's empirical formula

$$\underline{J} = \frac{q}{K} + bq^2$$

↑
gradient

where K is the hydraulic conductivity and b is a constant, depending on the porosity, average grain diameter, and shape factor but independent of fluid viscosity. The first term of the equation is linear (i.e., Darcy's Law) but the second term is nonlinear. This equation has been used to describe the non-Darcian flow.

In unsteady flow regimes, the fluid acceleration due to both the local and convective acceleration may be important.

$$F_P + F_\tau = ma = m \left(\underbrace{\frac{\partial q}{\partial t}}_{\substack{\text{not} \\ \text{zero}}} + q \underbrace{\frac{\partial q}{\partial x}}_{\substack{\text{not} \\ \text{zero}}} \right)$$

A modified Darcy's law has been developed to include these effects:

$$J = \frac{q}{K} + bq^2 + \underbrace{C \frac{\partial q}{\partial t}}_{\substack{\text{Local Acceleration} \\ \text{(transient term)}}} \quad (\text{Polubarinova-Kochina})$$

In generally, the effect of $\partial q / \partial t$ term is significant only during a fraction of a second. For all practical purposes, the "time-dependent" Darcy's Law has been often avoided.

Questions:

1. Is the cubic law valid for flow through a single fracture with large apertures? Why? If not, can you make any suggestions to improve the law?
2. Do you think that classical Darcy's law is valid for flow in many coastal aquifers made of coral reefs? Why not?
3. Can you use Darcy's law to describe ground water movement during an earthquake?