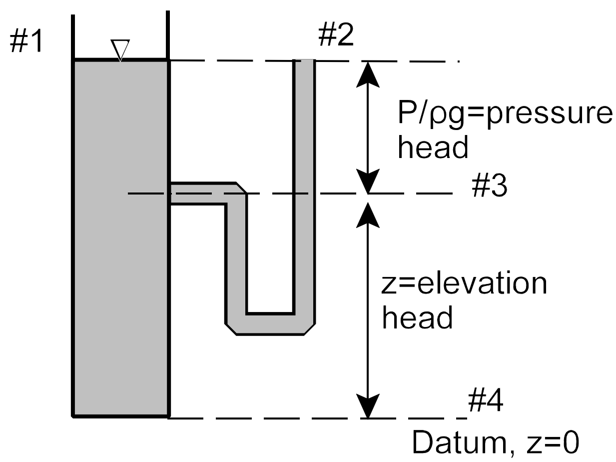
DARCY'S LAW FOR UNSATURATED POROUS MEDIA

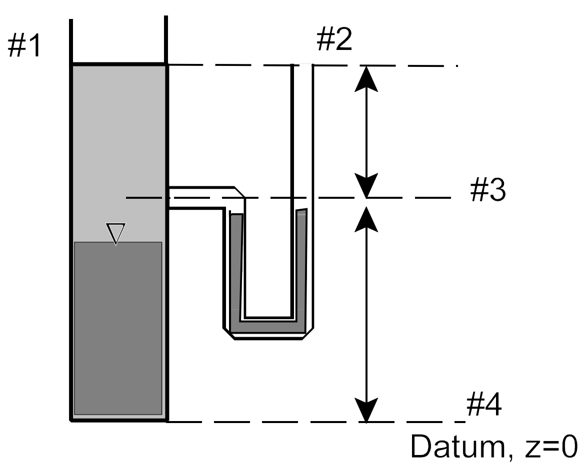
Before we discuss Darcy's Law for unsaturated media, let's first introduce the negative pressure concept using the following experiments.

CONCEPT OF THE NEGATIVE PRESSURE HEAD.

Consider a sand column filled with water and a manometer is installed on the side of the column (as shown in the figure below). The pressure head at location #3 is the distance between locations #3 and #2 and is positive. The elevation head is the distance measuring from locations #4 to #3 if the datum is chosen at location #4, where z is set to zero. What is the pressure head at #1 and #2?

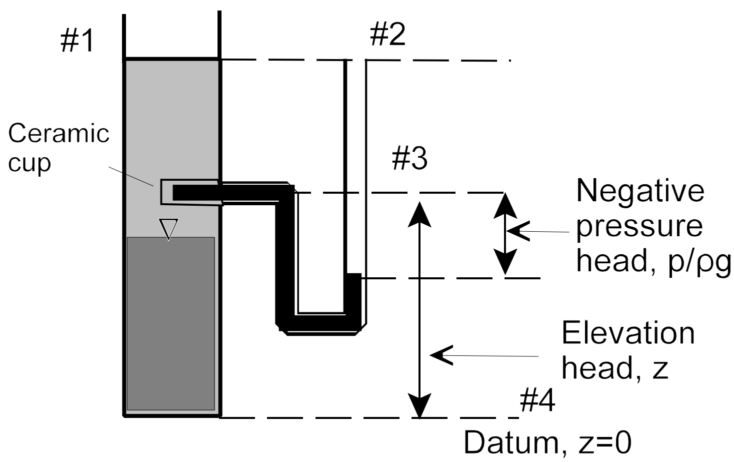


If the water level in the column drops below the location of the water manometer, the sand above the water table (indicated by ∇) is unsaturated. Air can easily enter the manometer and thus it can no longer serve as a pressure measurement device and cannot register the pressure in the soil.



To measure the pressure in the sand while it is unsaturated, the water manometer must be equipped with a ceramic cup (or membrane). Pores of the ceramic cup are extremely fine and are always saturated with water due to capillary forces. When the sand becomes unsaturated, the cup prohibits entrance of air into the manometer and thus it maintains a continuous water column. Such a manometer equipped with a ceramic cup is called a tensiometer.

As the water table in the sand column drops to the level of the ceramic cup, the water level in the tensiometer aligns with the water table, indicating the pressure head at the ceramic cup or in the tensiometer is zero or equal to the atmospheric pressure. Notice that when the sand drains, water in large pores in the sand flow first and remaining water mainly is confined to the small void spaces due to adhesive forces between water and sand particles. The adhesive forces and the cohesive forces between water particles create the capillary pressure in the sand, resisting external forces causing flow of water.



When the water table drops below the location of the ceramic cup, the capillary forces in the sand start to draw water from the ceramic cup and in turn, the tensiometer. After the capillary forces reach a new equilibrium with the ceramic cup, flow of water from the tensiometer to the sand stops. The water level in the tensiometer then represents the capillary pressure head in the soil. Since the water level in the tensiometer is now below the center of the ceramic cup, the pressure head of the unsaturated soil thus is negative, indicating its value is smaller than the atmospheric pressure (0).

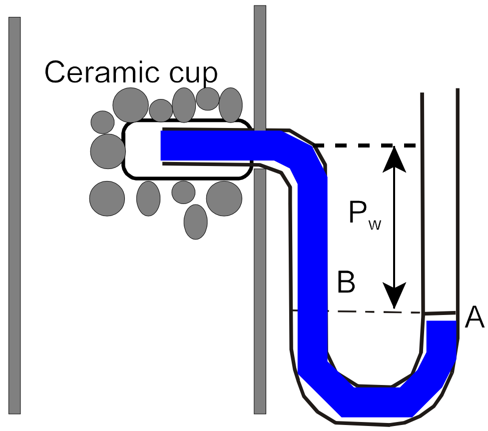
A blow-up of the tensiometer is shown in the next figure. Consider force balance in the tensiometer. The pressure at point A equals the atomospheric pressure, P_a , which must be equal to the pressure at point B. At point B, the upward pressure ought to be balanced by the downward force. The upward pressure is P_a and the downward pressure is the sum of the pressure due to the height of water column, P_w, and the total capillary pressure exerting on the ceramic cup, P_c. That is,

$$P_c + P_w = P_a$$

This leads to the definition of capillary pressure:

$$P_c = P_a - P_w$$

which states that the capillary pressure is the pressure of air (or the non-wetting fluid) minus the



pressure of water column (the wetting fluid). If we use P_a as the reference pressure, $P_a = 0$, the capillary pressure is then

$$P_c = -P_w$$

This capillary pressure represents the resultant capillary forces in all the voids adjacent to the ceramic cup due to cohesive and adhesive forces between water, air, and soil particles.

Subsequently, the total head (hydraulic head) thus becomes smaller than the elevation head. Note that many terminologies have been used for the negative pressure, namely, capillary pressure, soil water pressure, and matric potential. On the other hand, suction and tension represent the absolute value of the negative pressure.

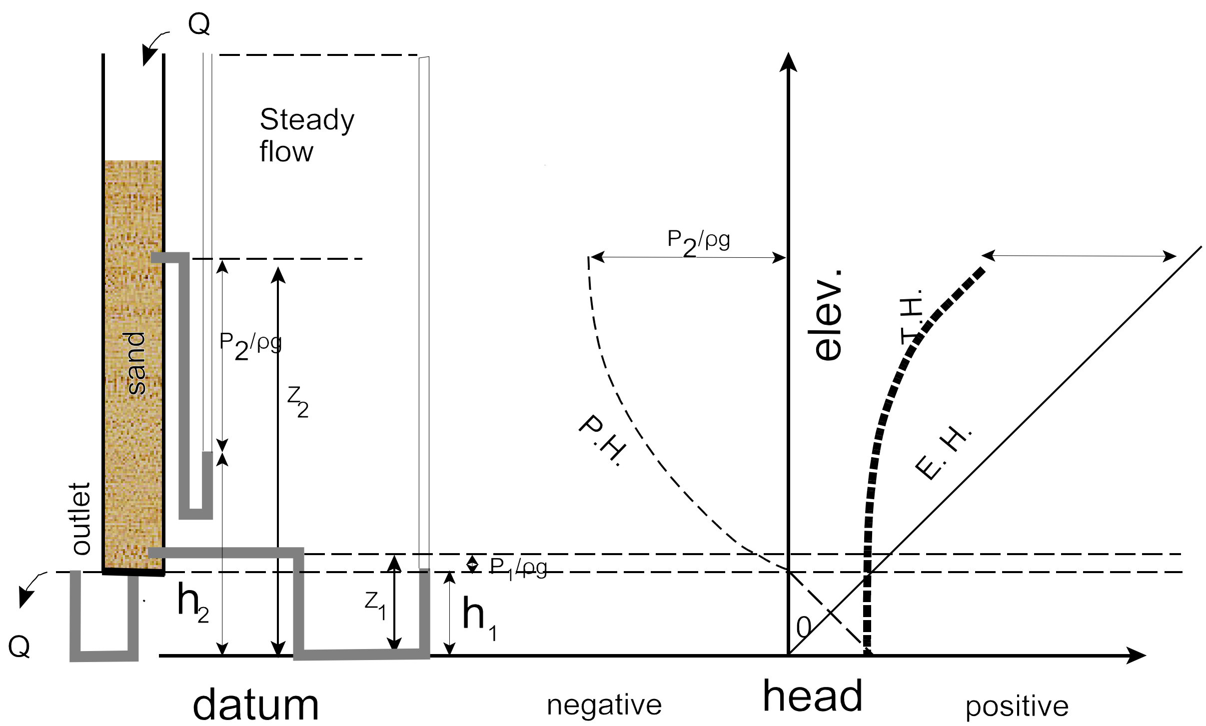
Questions:

a) Of course, the water level in the manometer takes time to reach an equilibrium at which you have a steady measurement of the capillary pressure. Can you speculate about the factors controlling the time required to reach the new equilibrium (the response time of a tensiometer)?

b) If the response time is large, how can you design a better tensiometer so that the response time is minimum?

c) Can a tensiometer measure positive pressures?

Now, consider a sand column experiment as shown above. If we first apply to the top of the



column a constant flow rate of water that equals the saturated hydraulic conductivity of the sand, we assume that there is no ponding at the top, and we assume that flow reaches a steady flow condition, then the sand is fully saturated and the flow is driven by the gravity only. Because the gravity is the only driving force, the pressure head distribution along the column must be uniform and the head equals the atmospheric pressure, P_a=0. Under this condition, the total head gradient is unity and we call that the flow is under a unit gradient condition.

If we now reduce the flow rate such that it is less than the saturated hydraulic conductivity, the sand becomes unsaturated because the amount of water supplied is not sufficient to fill all the pores in the sand. Therefore, capillary forces in the sand exist and the pressure of water is less than the atmospheric pressure. The water level in the tensiometer becomes lower than the elevation of the ceramic cup (see the figure below), a negative pressure head. Assuming that many tensiometers have been installed along the column, a plot of the tensiometer readings gives the pressure head distribution along the column (see Figure). Notice that the pressure at the outlet at the bottom of the column is equal to atmospheric pressure which is zero and it represents the water table.

Questions:

1. What do you observe from this experiment?
2. Are the pressure and total head distribution the same as in the saturated flow case? Why?
3. Is the hydraulic head gradient constant along the soil column?

WATER RELEASE CURVE OF UNSATURATED POROUS MEDIA

Definitions

The amount of water in soil can be defined in terms of volumetric water content or gravimetric water content. The volumetric water content is defined as the volume of liquid water per total volume of soil. That is,

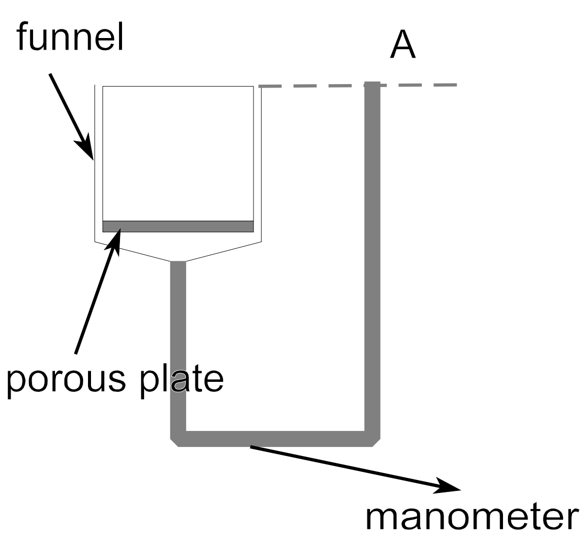
$$\theta_v = \frac{V_w}{V_T}$$

where V_w is the volume of water in the sample. V_T represents the total volume of the soil sample, including the volume of solids, V_s, and voids, V_v, in the sample. On the other hand, the gravimetric water content is defined as the mass of water per mass of dry soil. That is,

$$\theta_g = \frac{M_w}{M_s} = \frac{\rho_w V_w}{\rho_s V_s} = \frac{\rho_w V_w}{\rho_b V_T} = \frac{\rho_w}{\rho_b}\theta_v$$

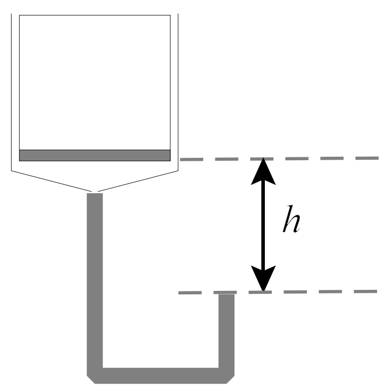
where ρ_w is the density of water and ρ_b is the bulk density.

Another term used frequently is the degree of saturation which is defined as the percentage of the porosity which is saturated.

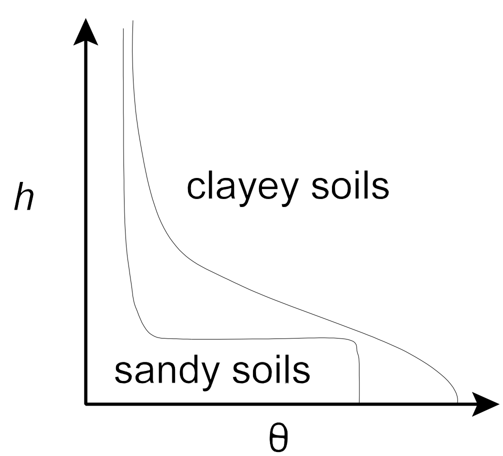


Water release (moisture retention) curves

From the previous experiment, we know that the pressure in unsaturated porous media is negative. Then, an immediate question one should ask is: is the negative pressure related to the degree of saturation or moisture content of the medium? To answer the question, let's consider a hanging column experiment. If we place a soil sample in a device as shown in the figure and the soil is fully saturated, the water level in the manometer should be level with the soil surface as shown in the figure. Now, if we lower the open end of the manometer (Point A in the figure), water will flow out from the manometer and the soil sample starts to drain. After a period of time, equilibrium will establish and the flow will stop due to capillary effect in the sand that holds water against the gravity force and suction created by lowering the manometer. Since the flow ceases, the pressure along the soil sample must be negatively hydrostatic (i.e., $h = -z+h_0$, where h is the pressure head, z is elevation from a given datum, h_0 is the pressure head at $z = 0$.). If we neglect the thickness of the porous plate, the pressure at the bottom of the soil sample, h, must be equal to the distance between the water level in the manometer and the porous plate, which is considered to be negative. The pressure in the sand should become more negative as the elevation increases. If the sample is small (say, 5 centimeters in height), we may assume that the entire sample is subject to the same negative pressure as the pressure at the bottom of the plate. If we know the porosity of the sample at saturation, and the amount of water discharged after lowering the manometer, the moisture remaining in the sample can be easily determined. If we successively lower the manometer and determine the moisture content of the sample, a plot of the relation between the negative pressure and the moisture content of the sample can be made (see Figure). This relation is called the water release curve or moisture and capillary pressure relationship of the sample.

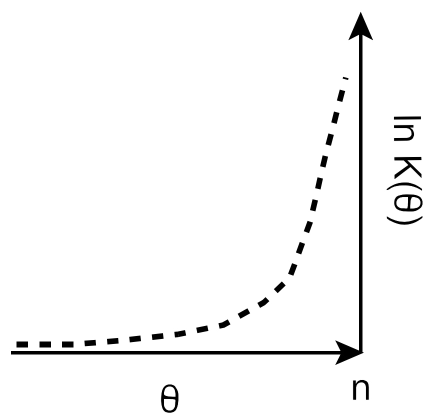


From the curve, it is clear that the moisture content of the sample decreases as the pressure becomes more negative. At saturation, the pressure is zero and the moisture content is equal to porosity. As the pressure becomes negative, the moisture content of the sample decreases rapidly due to the drainage of large pores. After drainage of the large pores, an increase in h does not reduce the moisture content significantly, due to strong capillary effects of the small pores. As a result, the water release curve of a porous medium often exhibits an S shape as shown in the figure. Generally speaking, the rate of reduction in moisture content as capillary pressure increases depends on the soil pore size distribution. For instance, well-sorted sands tend to have a narrow pore-size

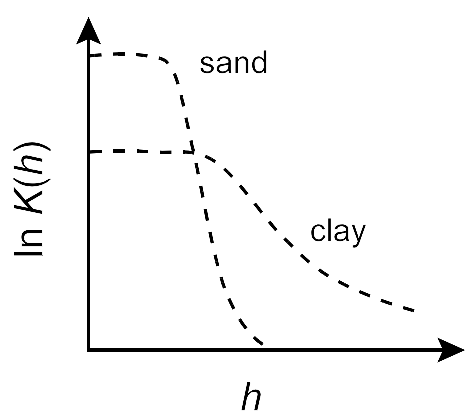


distribution (i.e., a large number of large pores and only a few small pores). Therefore, sandy materials tend to have a rapid reduction in water content as pressure increases. On the other hand, fine-textured materials such as clayey soils, they have a widespread pore-size distribution and the reduction in conductivity is thus gentle. Notice that the moisture content corresponds to $h = 0$, and is termed the saturated moisture content, θ_s. This saturated moisture content may not necessarily equal the porosity due to entrapped air. The moisture content that does not change significantly at very dry conditions regardless the pressure is called the residual moisture content, θ_r.

Unsaturated Hydraulic Conductivity Curve



Now, let's speculate about the behavior of the hydraulic conductivity of porous media under unsaturated conditions using our common sense without resorting to any fancy mathematics or physics. From the previous discussion about Poisuelle's law, we have learned that the saturation hydraulic conductivity is a function of the average radius of saturated pores, porosity, and tortuosity. To find out how the conductivity of unsaturated porous media would be different from their saturated hydraulic conductivity, we ask ourselves: how would the degree of saturation of a porous medium affect the radius of saturation pores, porosity, and tortuosity? If a fully saturated porous medium is being desaturated, large pores will drain first and thus the remaining water must flow in small pores. The average radius of the pores will be small and the average porosity (% of saturated voids) is reduced and the number of fully saturated voids decreases. Furthermore, the active flow channel that connects all the small pores becomes more tortuous. If all these are true, we can then say that the unsaturated hydraulic conductivity will be less than the saturated hydraulic conductivity. We may then conclude that the unsaturated hydraulic conductivity will decrease further with the further decrease in moisture content and it will approach almost zero after a threshold value of moisture content is reached. That is, porous



media with moisture content less than the threshold value virtually cannot transmit any significant amount of water because water attached to solids forms films that are isolated from each other. According to the above reasoning, it should be clear that the unsaturated hydraulic conductivity is a function of water content and its value decreases from saturated hydraulic conductivity as water content decreases (see the figure where ln$K(\theta)$ represents the natural logarithm of $K(\theta)$). Notice that since the moisture content is related to capillary pressure, the unsaturated hydraulic conductivity is also a function of capillary pressure and this relation is shown in the figure.

Questions:

What are the characteristics of the unsaturated hydraulic conductivity curves?
Why are $K(h)$ curves of sand and clay different as shown in the figure? Explain.

Certainly, one can measure the unsaturated conductivity of the porous medium at every capillary pressure head value and tabulate the $K(h)$ relationship in a table. However, for many practical purposes and convenience, mathematical models are often used to describe this relationship. One formula, frequently used to depict the unsaturated hydraulic conductivity and moisture release curves, is the exponential model (Gardner, 1958),

$$K(h) = K_s \exp(\beta h)$$
$$\theta(h) = (\theta_s - \theta_r)\exp(\beta h) + \theta_r$$

where K_s is the saturated hydraulic conductivity; β is the pore-size distribution parameter, [1/L] representing the rate of reduction in conductivity as the soil desaturates; θ_s is the saturated moisture content; and θ_r is the residual moisture content. The exponential model has been very popular owing to its simplicity and convenience in mathematical analysis. However, it fits the observed $K(h)$ or $\theta(h)$ data over only a limited range of pressure head values. Other widely-used models for $K(h)$, and $\theta(h)$ are those by Mualem (1976) and van Genuchten (1980):

$$K(h) = K_s \frac{\left(1-(\alpha h)^{n-1}\left[1+(\alpha h)^n\right]^{-m}\right)^2}{\left[1+(\alpha h)^n\right]^{m/2}}$$

$$\theta(h) = (\theta_s - \theta_r)\left[1+(\alpha h)^n\right]^{-m} + \theta_r$$

in which α [1/L], n [ ], and m [ ] are soil parameters and $m = 1 - 1/n$. These models are valid over a broader range of pressure values than the exponential model (van Genuchten and Nielson, 1985). Because of the use of these mathematical models for the functional relation between the unsaturated hydraulic conductivity, pressure head, and moisture content, soils can often be categorized by the parameters such as α, β, n, θ_s, θ_r, and K_s. For example, coarse textured soils are reported to have large values of α, n, β and K_s, and fine-textured soils to have small values (e.g., Stephens et al., 1987, see Table below). However, values of these parameters are not necessarily unique for a given geological medium due to hysteretic behavior in $K(h)$ and $\theta(h)$ relationships; these values can be different according to the wetting and drying histories of the medium.

In practice, water release curves of soils can be measured at less cost and effort than the unsaturated hydraulic conductivity. Parameters, α, β, n values of unsaturated hydraulic conductivity and water release curves are often conveniently assumed to be the same although they may be different (e.g., Yeh and Harvey, 1990). In addition, (2) and (3) are frequently used to extrapolate the conductivity to very dry conditions where a direct measurement of hydraulic conductivity is beyond our ability.

Three-Parameter Solution of *Mualem's* [1976] Model Using Imbibition Data *[Mualem and Da9an,* 1976]

Soil Type	Catalog Number	α (1/cm)	*N*	θ_r
Silt "Columbia"	2001	0.015511	1.7676	0.1369
Silt Mont Cenis (limon Silteaux)	2002	0.013647	1.3234	0.0000
Silt of Nave-Yaar	2003	0.072010	2.1969	0.3979
Rideau clay loam	3101	0.069118	2.0604	0.2863
Yolo light clay	3102	0.027000	1.6000	0.1800
Caribou silt loam	3301	0.047125	1.6981	0.2956
Grenville silt loam	3302	0.030702	1.2878	0.0326
Ida silt loam (> 15 cm)	3305	0.040000	1.2700	0.0000
Ida silt loam (0-15 cm)	3306	0.089975	1.1768	0.0000
Touched silt loam	3308	0.027302	3.5385	0.0993
Silt Loam G.E. 3	3310	0.004233	2.0594	0.1313
Gilat loam	3402	0.017000	2.3000	0.0846
Guelph loam	3407	0.073566	1.7844	0.2193
Rubicon sandy loam	3501	0.052321	1.8570	0.1388
Loamy Sand-Hamra Sharon	4004	0.018695	5.1537	0.1997
Plainfield sand (210-250 µ)	4101	0.045177	3.9979	0.0102
Plainfield sand (177-210 µ)	4102	0.038611	4.0409	0.0099
Plainfield sand (149-177 µ)	4103	0.032170	4.0570	0.0069
Plainfield sand (125-149 µ)	4104	0.024903	5.8327	0.0283
Plainfield sand (104-125 µ)	4105	0.022127	4.4446	0.0148
Sand	4106	0.094490	2.0422	0.0000
Sand	4107	0.060000	2.6400	0.0400
Del Norte fine sand	4108	0.016254	4.3600	0.0505
Oakley sand	4112	0.095194	2.0136	0.0255
G.E. 3 sand	4115	0.035965	4.4892	0.0409
Crab Creek sand	4117	0.118896	2.4506	0.0000
Sinai sand	4122	0.023803	5.3076	0.0326
Sand (50-500 µ)	4124	0.019116	4.6747	0.0693
Gravelly sand G.E. 9	4135	0.015048	2.8391	0.0793
Fine sand G.E. 2	4136	0.007192	3.8937	0.0608
Plainfield sand (0-25 cm)	4146	0.033730	3.8518	0.1133
Plainfield sand (25-60 cm)	4147	0.031813	4.1948	0.0724
Aggregated glass bead	5003	0.039748	6.4676	0.0983
Monodispersed glass bead	5004	0.036049	7.6171	0.0363

Source: Stephens et al., 23(12), *WRR*, 1987.

Moisture Capacity Curve. As illustrated earlier, the moisture content is a function of the capillary pressure and this relation is nonlinear. If we differentiate θ with respect to *h*, we will obtain a curve that represents the amount of change in moisture content per change in pressure at a given pressure value. This curve is important because it tells us the change in storage of the porous medium at a given pressure and it is called the moisture capacity curve of the porous medium. That is,

$$C(h) = \left.\frac{d\theta}{dh}\right|_h = \left.\frac{\Delta\theta}{\Delta h}\right|_h$$

The vertical line in the equation indicates that the term is evaluated at the given *h*. The unit of the moisture capacity term is [1/*L*].

Question:

1. Can you draw *C*(*h*) curves for sand and clay? Why are they different?
2. What is the equivalent term for moisture capacity in the saturated porous media?

Darcy's Law (Buckingham-Darcy's Law) for Unsaturated Porous Media

Based on what we have discussed, we may be able to formulate Darcy's Law for unsaturated porous media for one-dimensional vertical flow.

$$q = -K(h)\frac{d\phi}{dz} = -K(h)\frac{d(h+z)}{dz} = -K(h)\left[\frac{dh}{dz}+1\right]$$

where ϕ is the hydraulic head which is equal to the sum of the pressure head, *h*, and the elevation head, *z*. *dh*/*dz* represents the pressure gradient and *dz*/*dz* = 1 presents the gravity (or elevation gradient). Note that the pressure head gradient may work against the gravity (can you give me an example?) Now, what is Darcy's Law for horizontal unsaturated flow?

Notice the difference in Darcy's Law for flow in saturated and unsaturated porous media. Under saturated conditions, conductivity is a function of medium and fluid properties. It is also a function of capillary pressure (or moisture content) under unsaturated conditions. Its dependence upon pressure makes the equation nonlinear.

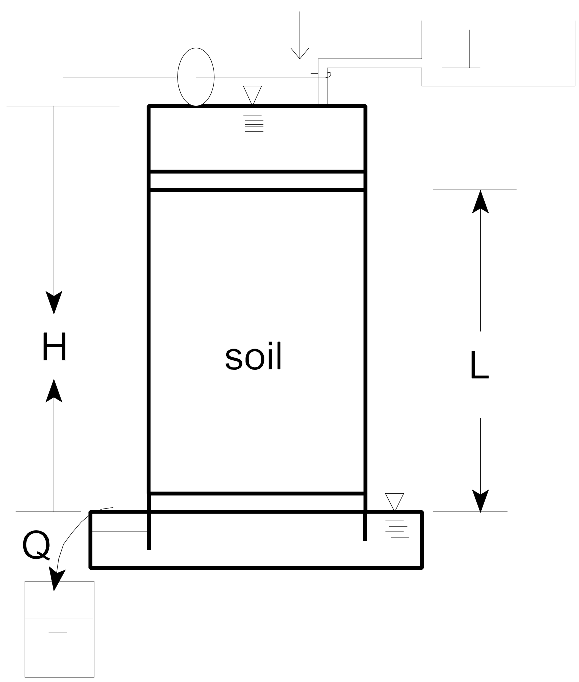
An important point that needs discussion is that at a given pressure the specific discharge in the unsaturated medium is assumed to be linearly proportional to the hydraulic gradient as in the saturated medium. No experiment has been conducted to verify this assumption as far as I have known. This may be attributed to the difficult in designing experiments for specifically verifying this assumption. In spite of this fact, this convenient assumption seems to work well for many practical purposes and hydrologists have taken it for granted.

Questions:

1. Can you think of any factors that can invalidate Darcy's Law for flow in unsaturated media?
2. How does the dependence of conductivity on pressure affect pressure distribution in unsaturated porous media? Can you explain the difference in the pressure head distributions under saturated and unsaturated conditions in the soil column experiments being discussed previously?
3. Can you design an experiment to test Darcy's Law for unsaturated flow?

Laboratory Measurements of Saturated Hydraulic Conductivity

1. Constant Head Permeameter



use Darcy's Law

$$Q = KA\frac{H}{L} \qquad K = \frac{QL}{AH}$$

$$Q = \frac{V}{t}$$ V = volume of water collected in a beaker at Δt

Cautions:

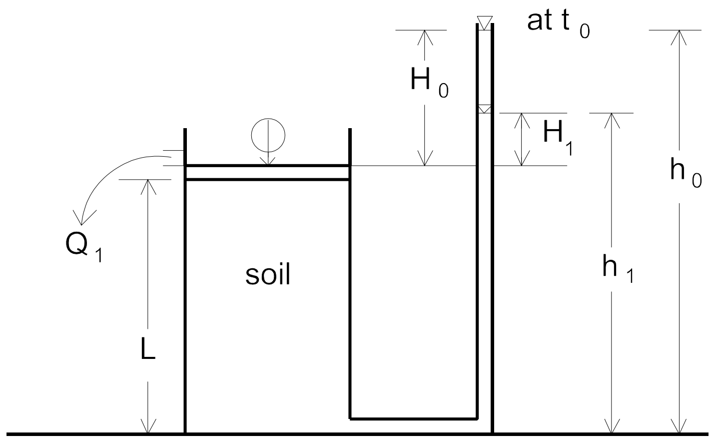
1) Use de-aired water to avoid entrapped air in the system.
2) Fully saturate the sample from below as they are emplaced.
3) Always take many different Q measurements under different gradients and then carry out a linear regression between Q's and gradients to obtain the slope which is the average conductivity.

Questions:

1. How does a fine porous plate, used to support the porous medium in a soil column during a conductivity test, affect your conductivity measurement?

2. A prominent scientist proposes to conduct measurement of mixed components of the hydraulic conductivity tensor. Can you think of any reason to rebut this proposal by the scholar?

2. Falling-Head permeameter



(2)

Discharge at point (1),

$$Q_1 = KA\frac{(\phi_2 - \phi_1)}{L}$$

$$\phi_1 = \frac{P_1}{\rho g} + Z_1 = L$$

$$0,\ P_1 = P_a = 0$$

$$\phi_2 = \frac{P_2}{\rho g} + Z_2 = h(t)$$

$$Q_1 = KA\left(\frac{h(t) - L}{L}\right) \qquad --Q$$

where A is the cross-sectional area of the permeameter.

Discharge from the manometer (2)

$$Q_2 = -a\frac{dh}{dt} \qquad --(2)$$ (neg.) decrease with time

assuming a linear relationship between Q and $\frac{dh}{dt}$, where a is the cross-sectional area of the manometer.

$Q_1 = Q_2 \quad \rightarrow$ Assuming no storage effect instantaneous

From (1) and (2):

$$-a\frac{dh}{dt} = KA\frac{(h(t) - L)}{L}$$

$$\int_{h_o}^{h_1} \frac{dh}{(h - L)} = \frac{KA}{aL}\int_{t_o}^{t_1} dt$$

$$\ln(h - L)\Big|_{h_o}^{h_1} = \frac{-KA}{aL}(t_1 - t_o)$$

$$\Rightarrow \ln\left[\frac{(h_1 - L)}{(h_o - L)}\right] = \frac{-KA}{aL}t$$

or

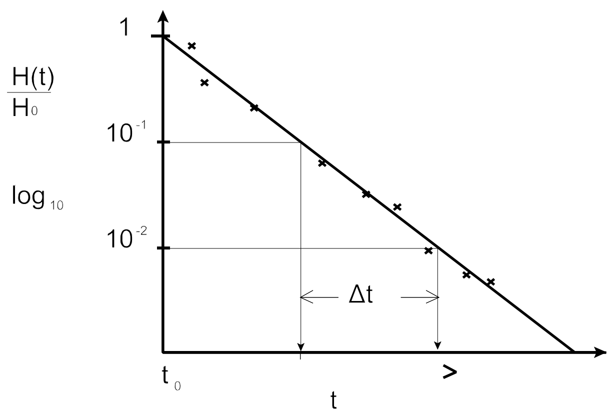
$$\ln\left(\frac{H_1}{H_o}\right) = \frac{-KA}{aL}t_1$$

or

$$\ln\left(\frac{H(t)}{H_o}\right) = \frac{-KA}{aL}t \qquad \ln = 2.3\log_{10}$$

$$q = -K(h)\frac{d(h+z)}{dz} = -K(h)[\frac{dh}{dz}+1] \qquad \ln 2 = 0.69 \qquad \log 2 = 0.3$$

$$\log_{10}\left(\frac{H(t)}{H_o}\right) = -\frac{1}{2.3}\frac{KA}{aL}t$$



choose 1 log cycle

$$1 = +2.3\frac{KA}{aL}\Delta t$$

$$K = \frac{2.3aL}{A\Delta t}$$

Questions:

1. The falling head permeameter assumes an instantaneous outflow, neglecting the storage effect. How would you design your apparatus to avoid impacts of this assumption?
2. Will this analysis be much different from the mathematical analysis of a slug test result?

Measurement of Unsaturated Hydraulic Conductivity

A. Steady Flow approach:

The following methods for measuring unsaturated hydraulic conductivity are based on

1. Non-unit gradient approach:

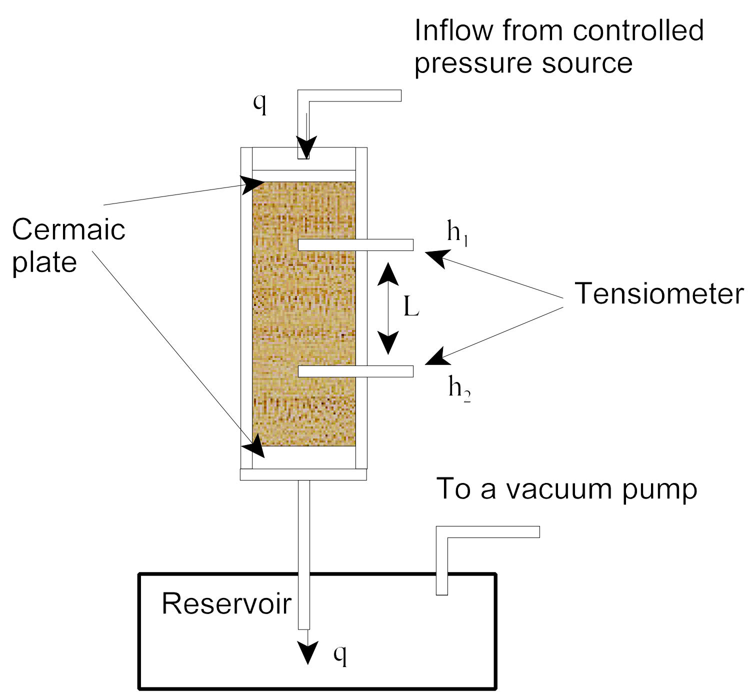
Consider steady unsaturated flow in a soil column enclosed by two ceramic plates. The bottom plate is connected to a closed reservoir, under a given vacuum. The top plate is connected to a water source that is also under some vacuum. Two tensiometers, separated by a short distance L, are installed along the column to measure the pressure heads h_1 and h_2. If the specific discharge q is known, the unsaturated hydraulic conductivity can then be calculated by

$$K(\bar{h}) = -\frac{q}{(\frac{h_1 - h_2}{L}) + 1}$$

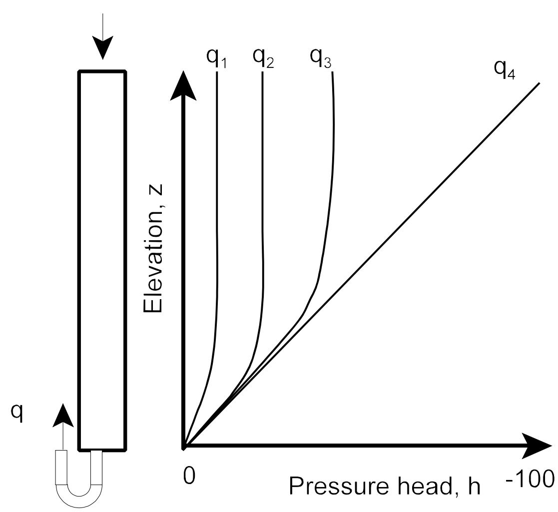
where $\bar{h} = (h_1 + h_2)/2$. By varying the specific flux and using this approach, one can obtain many pairs of K and h values and thus, a conductivity/pressure relationship can be constructed. In this approach, the two tensiometers must be installed closely so that the pressure head variation between the two points is approximately linear and pressure difference is small so that the average of the two pressure heads is a good approximation of the pressure for the conductivity.

2. Unit Gradient approach:

A short column approach: Using the apparatus shown above, if the pressure values at point 1 and point 2 are adjusted to reach the same pressure and the flow is steady, from Darcy's Law, we see that $q = -K(h)$. By varying the q value to reach different steady state conditions, one can obtain many $K(h)$ values. While the method is conceptually simple, it is difficult to adjust fluxes and pressures at bottom and top to reach a unit gradient and the steady flow condition.



An alternative to the short column approach is the long column approach. That is, it is possible to obtain the unit gradient condition in the upper part of a long column during steady downward flow to a "water table" near the base of the column. This can be done without any ceramic plates at the top and bottom and without any adjustment other than that necessary to control the inflow rates at the top of the column. To illustrate the approach, consider a long column instrumented with several tensiometers as shown in the figure so that the pressure profile along the column can be monitored. Let a constant flux that is slightly smaller than the saturated hydraulic conductivity be applied to the column. Once the steady flow establishes, the pressure head profile is recorded. By varying the inflow rate, several pressure head profiles of steady flow can be obtained as illustrated in the figure. Notice that all the pressure profiles near the upper portion of the column exhibit essentially uniform pressure distributions, indicating unit gradient conditions. Therefore, the uniform pressure head and the known inflow rate constitute a pair of the unsaturated hydraulic conductivity and the pressure head. Many pairs of the hydraulic conductivity and the pressure head data sets then produce the unsaturated conductivity curve.



B. Transient Approach

To be discussed later.

1. Instantaneous profile method

2. Outflow method.