§ **Transient flow to a well in a confined aquifer** (535-5b1.wpd)

Is there any steady-state flow in the real world? If there is one, the world must be very boring or you must live in the twilight zone where digit heads live. So, in general we are not interested in a steady-state solution. Let's proceed formally to the analysis of transient radial flow in a confined aquifer.

First, we will assume that the aquifer is confined with infinite lateral extent, isotropic, and homogeneous, and the discharging well is fully penetrating, screened over the entire thickness of the aquifer, and discharging at a constant rate, Q.

P.D.E.

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t}$$

B.C.

$$1.\ \lim_{r\to 0}\left(r\frac{\partial h}{\partial r}\right) = \frac{Q}{2\pi T}$$

$$2.\ h(\infty,t) = h_0 \quad \text{at} \quad \text{all} \ \ t\text{'s}$$

I.C.

$$h(r,0) = h_0 \qquad \text{at} \quad \text{all} \ \ r\text{'s}$$

These equations can be written in terms of drawdown

$$\delta(r,t) = h_0 - h(r,t)$$

$$\frac{\partial^2 \delta}{\partial r^2} + \frac{1}{r}\frac{\partial \delta}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}$$

B.C.

$$1. \lim_{r\to 0}\left(r\frac{\partial \delta}{\partial r}\right) = \frac{Q}{2\pi r}$$

2. $\delta(\infty,t) = 0$ at all t's

3. $\delta(r,0) = 0$ at all r's

Theis solution (1935) using Boltzman's transformation

$$u = \frac{r^2 S}{4Tt} \qquad\qquad \frac{\partial u}{\partial r} = \frac{2}{4}\frac{rS}{Tt} = \frac{2u}{r}$$

$$\frac{\partial s}{\partial r} = \frac{ds}{du}\frac{\partial u}{\partial r} = \frac{rS}{2Tt}\frac{ds}{du} \qquad \text{----a}$$

$$\frac{\partial^2 s}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial s}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{rs}{2Tt}\frac{ds}{du}\right)$$

$$= \frac{S}{2Tt}\frac{ds}{du} + \frac{rs}{2Tt}\frac{\partial}{\partial r}\left(\frac{ds}{du}\right)$$

$$= \frac{S}{2Tt}\frac{ds}{du} + \frac{rs}{2Tt}\frac{d\left(\frac{ds}{du}\right)}{du}\cdot\left(\frac{\partial u}{\partial r}\right)$$

$$= \frac{S}{2Tt}\frac{ds}{du} + \frac{rs}{2Tt}\left(\frac{rs}{2Tt}\right)\cdot\frac{d^2 s}{du^2}$$

$$\frac{\partial^2 s}{\partial r^2} = \frac{S}{2Tt}\frac{ds}{du} + \frac{r^2 s^2}{4T^2 t^2}\frac{d^2 s}{du^2} \qquad \text{----b}$$

$$\frac{\partial s}{\partial t} = \frac{ds}{du}\frac{\partial u}{\partial t} = -\frac{r^2 S}{4Tt^2}\frac{ds}{du} \qquad \text{----c}$$

Substitution of a, b, and c into eq. (1) results in:

$$4u^2\frac{d^2 s}{du^2} + \left(4u^2 + 4u\right)\frac{ds}{du} = 0$$

or

$$u\frac{d^2 s}{du^2} + (u+1)\frac{ds}{du} = 0$$

We have reduced the nasty P.D.E. to a 2nd order O.D.E.

$$u \to \infty$$
$$s(\infty) = 0$$
$$\lim_{u\to 0} u\frac{ds}{du} = \frac{Q}{4\pi T}$$

$$\frac{d}{du}\left(u\frac{\frac{ds}{du}}{\Phi}\right) + u\frac{\frac{ds}{du}}{\Phi} = 0 \qquad \text{Let} \quad \Phi = u\frac{ds}{du}$$

Solution

$$\frac{d\Phi}{du} = -\Phi$$
$$\int \frac{d\Phi}{\Phi} = -\int du$$
$$\ln\Phi = -u + c \qquad \Phi = e^{-u+c} = e^{-u}\cdot e^{c}$$
$$u\frac{ds}{du} = c_1 e^{-u} \quad \to \quad \lim_{r\to 0} u \to 0 \qquad u = \frac{r^2 S}{4Tt}$$
$$\lim_{u\to 0}\left[u\frac{ds}{du}\right] = c_1\cdot 1 = \frac{Q}{4\pi T}$$
$$\therefore \frac{ds}{du} = \frac{c_1}{u}e^{-u} = \frac{Q}{4\pi T}\frac{e^{-u}}{u}$$

The required solution is obtained by integrating ds/du:

$$s = \frac{Q}{4\pi T}\left[\int_u^{-\infty}\frac{e^{-x}}{x}dx\right] \qquad \to \text{exponential integral}$$

or

$$s(r,t) = \frac{Q}{4\pi T}W(u) \qquad \text{where } W(u) = \int_u^{\infty}\frac{e-x}{x}dx$$

This is called the **Theis Equation** and the function, $W(u)$, is called the well-function for non-leaky
uniform confined aquifers which can be expressed in an infinite series

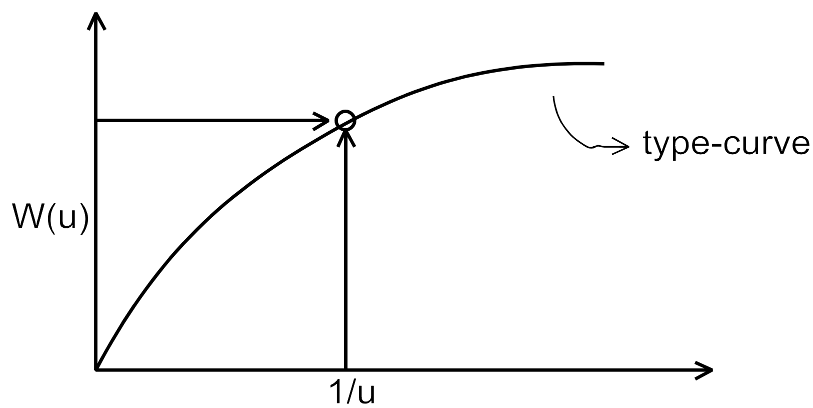
$$W(u) = \left[-0.5772 - \ln u + u - \frac{u^2}{2.2!} + \frac{u^3}{3.3!} + \cdots \right]$$

See Table for $W(u)$ vs. u or use a numerical approximation of $W(u)$

§ Graphical Solution Technique

$$s(r,t) = \frac{Q}{4\pi T} W(u)$$

(1) plot $W(u)$ vs. $1/u$ on log-log paper



(2) Calculate

$$u = \frac{r^2 S}{4Tt} \rightarrow \frac{1}{u}$$

(3) Find $W(u)$ then $S(r, t)$

Polynomial and Rational Approximation for $W(u)$

from: Abramowitz and Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1964, p. 231

$$W(u) = E_1(u)$$

§ Dimensionless Drawdown

$$s(r,t) = \frac{Q}{4\pi T} W(u), \qquad t = \frac{r^2 S}{4T}\left(\frac{1}{u}\right)$$

The drawdown and time can be rewritten as dimensionless drawdown and time. That is,

$$s^* = \frac{s(r,t)4\pi T}{Q} = W(u), \qquad t^* = \frac{1}{u} = \frac{4Tt}{r^2 S}$$

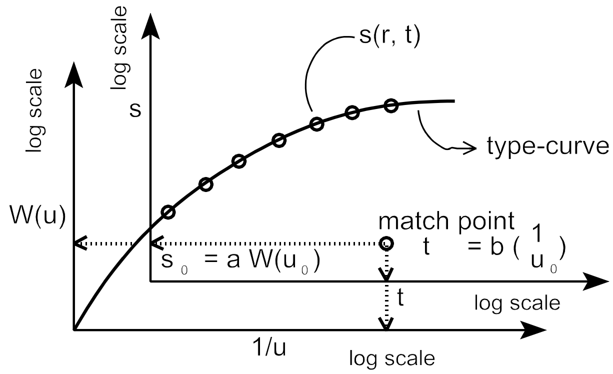
It is clear that the well function and $1/u$ are dimensionless drawdown and time, respectively. In other words, the real drawdown is linearly proportional to $W(u)$ and t is linearly proportional to $1/u$.

$$s = aW(u) \qquad t = b\left(\frac{1}{u}\right)$$

If we take the log of the above equations, we have

$$\log s = \log a + \log W(u) \qquad \log t = \log b + \log(1/u)$$

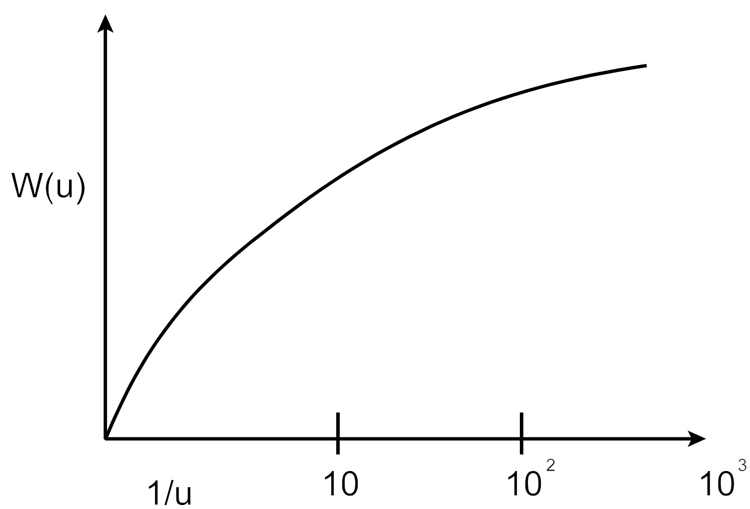
This implies that the plot of s vs. t on log-log paper is displaced from the plot of $W(u)$ vs. $1/u$ by an amount $\log a$ on the drawdown axis and $\log b$ on the time axis. This leads to the idea of type-curve matching.



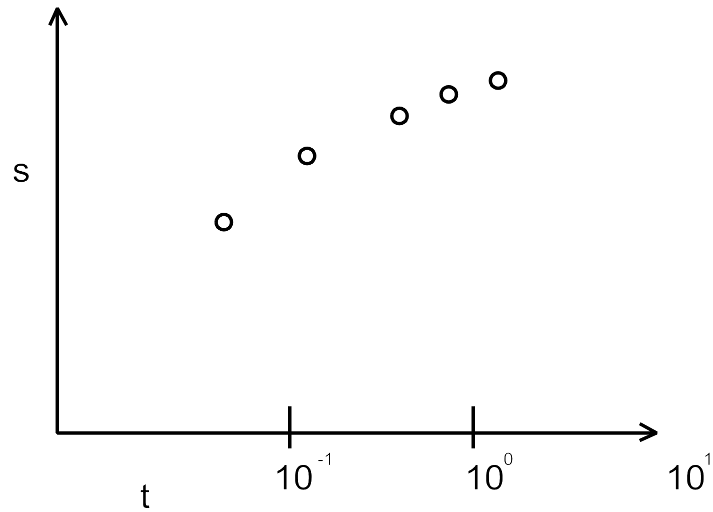
§ Parameter Estimation

Type-Curve Method

1) Plot the type-curve on log-log paper



2) Plot data on transparent paper with the same length of each cycle as that for the type curve.



3) Overlap the data sheet onto the type curve, holding the coordinate axes of the sheet and the type curve in such a way that the data best fit the type curve.
4) A common point, the match point, arbitrarily chosen on the overlapping part of the curves determines mutual values of:

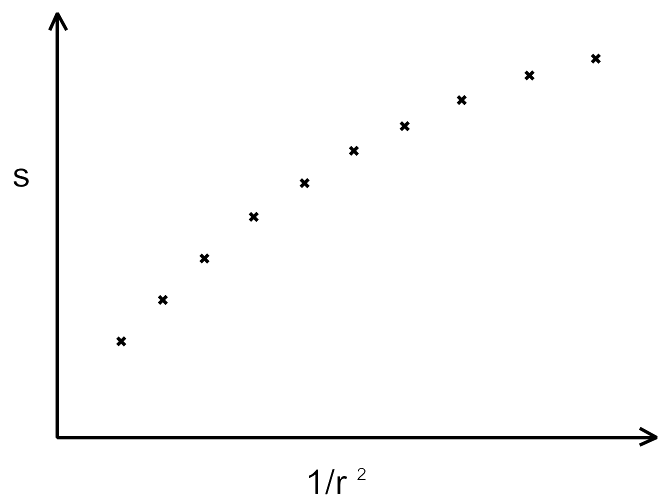
$$s \rightarrow W(u) \qquad \& \qquad t \rightarrow \frac{1}{u}$$

5) Determine T & S

$$T = \frac{Q}{4\pi s} W(u)$$
$$S = \frac{4Ttu}{r^2}$$

For any consistent units.

* The same procedure can be applied to s - r (drawdown ~ distance) data:


s
1/r 2

4) <u>The Jacob straight-line method</u>

Recall

$$W(u) = -0.5772 - \ln u + u - \frac{u^2}{2.2!} + \frac{u^3}{3.3!} - \frac{u^4}{4.4!} + \cdots$$

For small values of u, say $u \leq 0.01$, the infinite series can be approximated by its first two terms, i.e.,

$$s = \frac{Q}{4\pi T}(-0.5772 - \ln u)$$

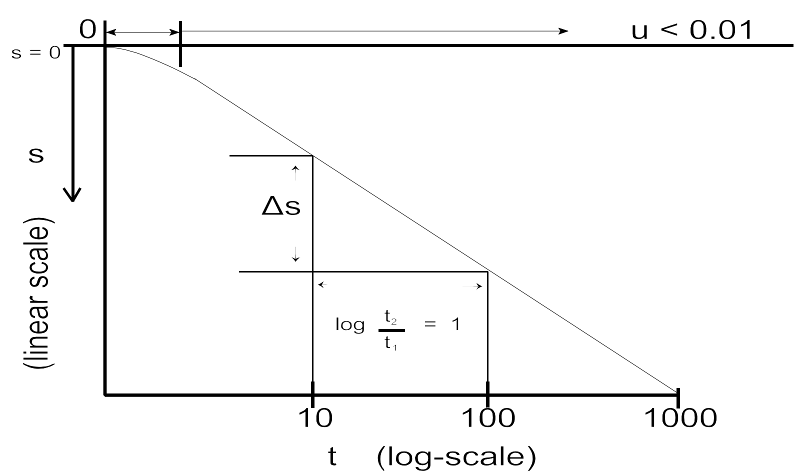
The error involved in this approximation is less than 6%

$$s = \frac{Q}{4\pi T}\left(\ln\frac{4Tt}{r^2 S} - \ln 1.78\right) = \frac{Q}{4\pi T}\ln\left[\frac{2.25Tt}{r^2 S}\right]$$

or

$$s = \frac{0.183Q}{T}\log\left[\frac{2.25Tt}{r^2 S}\right] \qquad \ln = 2.303\log_{10}$$

since all the log-papers are using $\log_{10}$. This equation implies that s is linearly proportional to log t. That is, if s is plotted on a linear scale and t is on the log-scale, the drawdown time data will follow a straight line when $u \leq 0.01$ or r is small or t is large.



Further, for $t = t_1$, the drawdown at a given distance r is

$$s_1 = \frac{0.183Q}{T} \log\left[\frac{2.25Tt_1}{r^2 S}\right]$$

For $t = t_2$

$$s_2 = \frac{0.183Q}{T} \log\left[\frac{2.25Tt_2}{r^2 S}\right] \qquad \text{at } r$$

The difference of the drawdowns at the two times is

$$\therefore \Delta s = s_2 - s_1 = \frac{0.183Q}{T} \log\left[\frac{t_2}{t_1}\right]$$

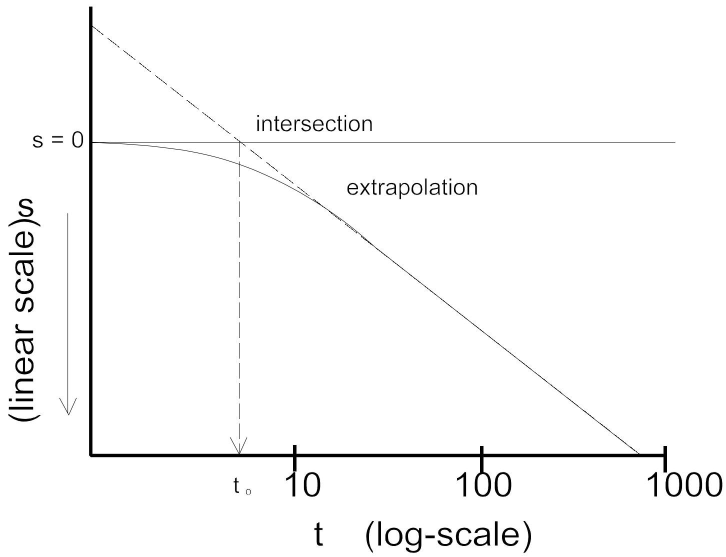
$$\Delta s = \frac{0.183Q}{T} \log\left[\frac{100}{10}\right] = \frac{0.183Q}{T}$$

and T can be estimated from

$$T = \frac{0.183Q}{\Delta s}$$

While the Jacob's straight line method provides you a convenient approach to estimate the transmissivity, it also tells you something that is important about the late-time behavior of the aquifer. That is, the rate of change in drawdown at the late time is mainly controlled by the transmissivity alone. A practical implication for aquifer test analysis is that one should use the late time data to estimate the transmissivity and then use the early time to estimate the storage coefficient.

Now, if $Q \neq 0$, $T \neq \infty$ we extrapolate the straight line to the point where the drawdown is zero



The straight line will intercept the line representing $s = 0$ and it is described by the equation

$$s = \frac{0.183Q}{T}\log\left(\frac{2.25Tt}{r^2 S}\right)$$

$$\Rightarrow \log\left(\frac{2.25Tt_0}{r^2 S}\right) = 0$$

$$\frac{2.25Tt_0}{r^2 S} = 1$$

$$S = \frac{2.25Tt_0}{r^2}$$

where t_0 is the position at which the $s = 0$ line intercepts the s - $\log t$ straight line.

§ Straight-line plot of *s* vs. log *r*

Two observation wells at distances r_1 and r_2 and drawdown of the two wells are s_1 and s_2 respectively.

$$s_1 = \frac{Q}{4\pi T}(-0.5772 - \ln u_1)$$
$$s_2 = \frac{Q}{4\pi T}(-0.5772 - \ln u_2)$$

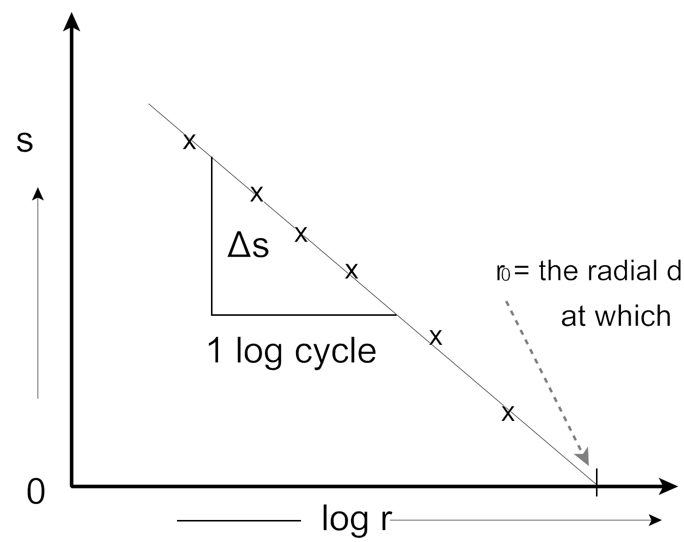
$$\therefore\ s_1 - s_2 = \frac{Q}{4\pi T}\ln\frac{u_2}{u_1}$$

$$\Delta s = \frac{Q}{4\pi T}\ln\frac{\left(\frac{r_2^2 S}{4Tt}\right)}{\left(\frac{r_1^2 S}{4Tt}\right)} = \frac{Q}{4\pi T}\ln\left(\frac{r_2^2}{r_1^2}\right)$$
$$\Delta s = \frac{Q}{2\pi T}\ln\left(\frac{r_2}{r_1}\right)$$

$$\Rightarrow\ \Delta s = \frac{2.303Q}{2\pi T}\log\left(\frac{r_2}{r_1}\right)$$

If we find the difference in drawdown between two points , separated by on log cycle, we have

$\Delta s = \dfrac{2.303Q}{2\pi T}$. Therefore, the estimate of the transmissvity is $T = \dfrac{2.303Q}{2\pi \Delta s}$



Now, we know that at r_0, $s = 0$

$$\Rightarrow 0.5772 = -\ln u = \ln u^{-1} = \ln\left(\frac{1}{u}\right)$$

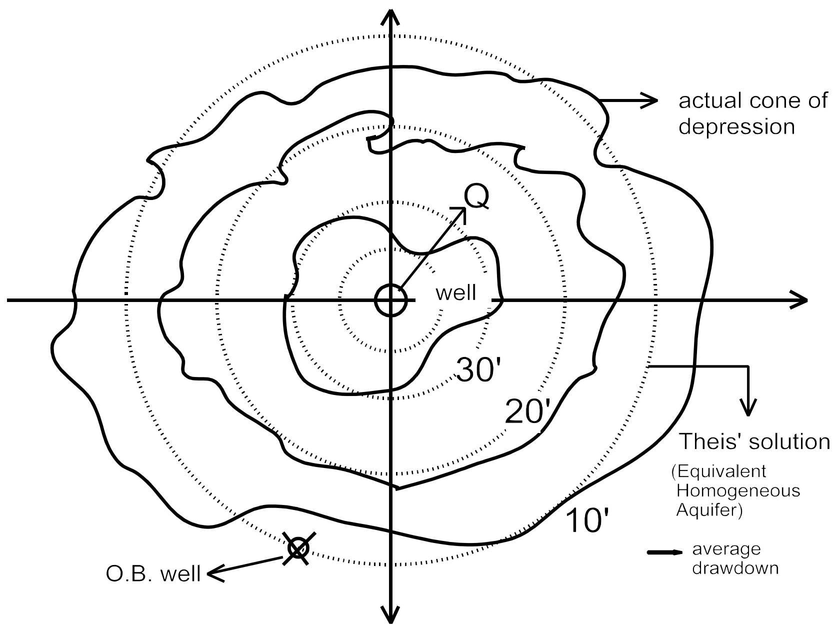
$$\frac{4Tt}{r_0^2 s} = e^{0.5772} = 1.78$$

$$S = \frac{2.25Tt}{r_0^2}$$

§ Real World Problems

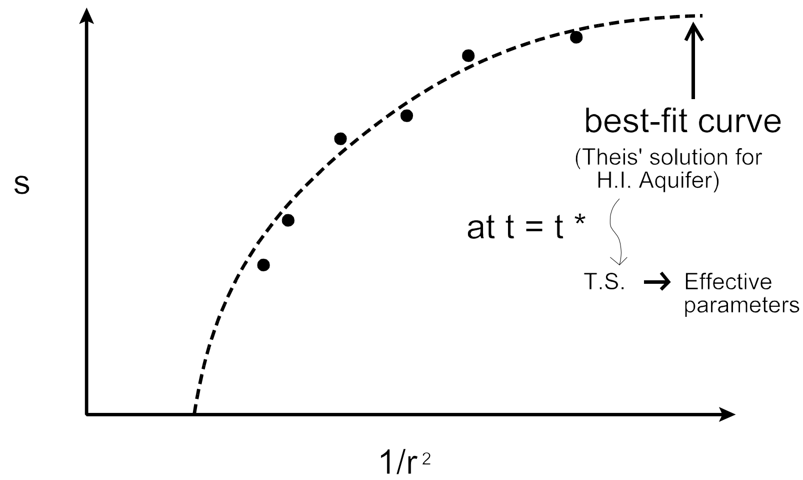
Aquifers are inherently heterogeneous. However, the Theis solution assumes aquifer homogeneity. So, is it logical to plug in the observed drawdown in a real aquifer to the Theis solution to back out the transmissivity or storage coefficient? This is equivalent to saying that Jim Yeh has a Rolles Royce and so every Chinese has one. Come on! use your common sense. More precisely, do you think the drawdown in an observation well tapping into a clay lens represents the change of water level in the other locations of the aquifer, even at the same radial distance? I do not know why groundwater hydrologists are so ideological. They must have learned groundwater hydrology from David Koresh in Waco, Texas. Did you?

See LaDon Jones, Tacy Lemar, and Chin-Ta Tsai, Results of two pumping tests in Wisconsin age weathered till in Iowa, Vol. 30, No. 4, 529-538, Ground Water, July-August 1992.



Comments

1. Estimating T & S based on s - t curve (data) from one O.B. well is inappropriate.
2. use many wells.
3. use distance-drawdown data



Wait! Can you trust what I just said? Again, if you have some common sense you should have asked why results of numerous aquifer tests demonstrated that the estimated transmissivity values using the ideological approach are reasonable? So tell me WHY?

Anisotropic Aquifers

Governing P.D.E.

$$T_x \frac{\partial^2 s}{\partial x^2} + 2T_{xy} \frac{\partial^2 s}{\partial x \partial y} + T_y \frac{\partial^2 s}{\partial y^2} = S \frac{\partial s}{\partial t} \qquad \text{----(1)}$$

- Coordinate transformation $\rightarrow$ x - y align with the principal directions

$$T_{x'} \frac{\partial^2 s}{\partial x'^2} + T_{y'} \frac{\partial^2 s}{\partial y'^2} = S \frac{\partial s}{\partial t} \qquad \text{----(2)}$$

- convert it to isotropic system

$$x'' = \sqrt{Ty'}x', \quad y'' = \sqrt{Tx'}y'$$

$$\left(\sqrt{T_{x'}T_{y'}}\right)\left(\frac{\partial^2 s}{\partial x''^2} + \frac{\partial^2 s}{2\partial y''^2}\right) = \left(\frac{S}{\sqrt{T_{x'}T_{y'}}}\right)\frac{\partial s}{\partial t}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$T_e\left(\frac{\partial^2 s}{\partial x''^2} + \frac{\partial^2 s}{2\partial y''^2}\right) \quad = \quad S_e \frac{\partial s}{\partial t}$$

The aquifer is isotropic in x'' and y'' coordinate system.

$\rightarrow$ Change (3) to radial coordinate (polar Co.)

$$\frac{\partial^2 s}{\partial r_e^2} + \frac{1}{r_e}\frac{\partial s}{\partial r_e} = \frac{S_e}{T_e}\frac{\partial s}{\partial r} \qquad \text{----(4)}$$

where

$$r_e^2 = x''^2 + y''^2 = T_{y'x'^2} + T_{x'y'^2}$$

Note

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

Then, we have

$$r_e^2 = T\cdot(x\cos\theta + y\sin\theta)^2 + T_{y'}\cdot(-x\sin\theta + y\cos\theta)^2$$

$$r_e^2 = T_x y^2 + T_y x^2 - 2T_{xy}xy$$

and

$$T_e = \sqrt{T_{x'}T_{y'}} \qquad \text{or} \qquad T_e^2 = T_{x'}T_{y'}$$

recall

$$T_{x'} = \frac{T_x + T_y}{2} + \left[\left(\frac{T_x - T_y}{2}\right)^2 + T_{xy}^2\right]^{\frac{1}{2}}$$

$$T_{y'} = \frac{T_x + T_y}{2} - \left[\left(\frac{T_x - T_y}{2}\right)^2 + T_{xy}^2\right]^{\frac{1}{2}}$$

$$T_e^2 = T_x T_y - T_{xy}^2$$

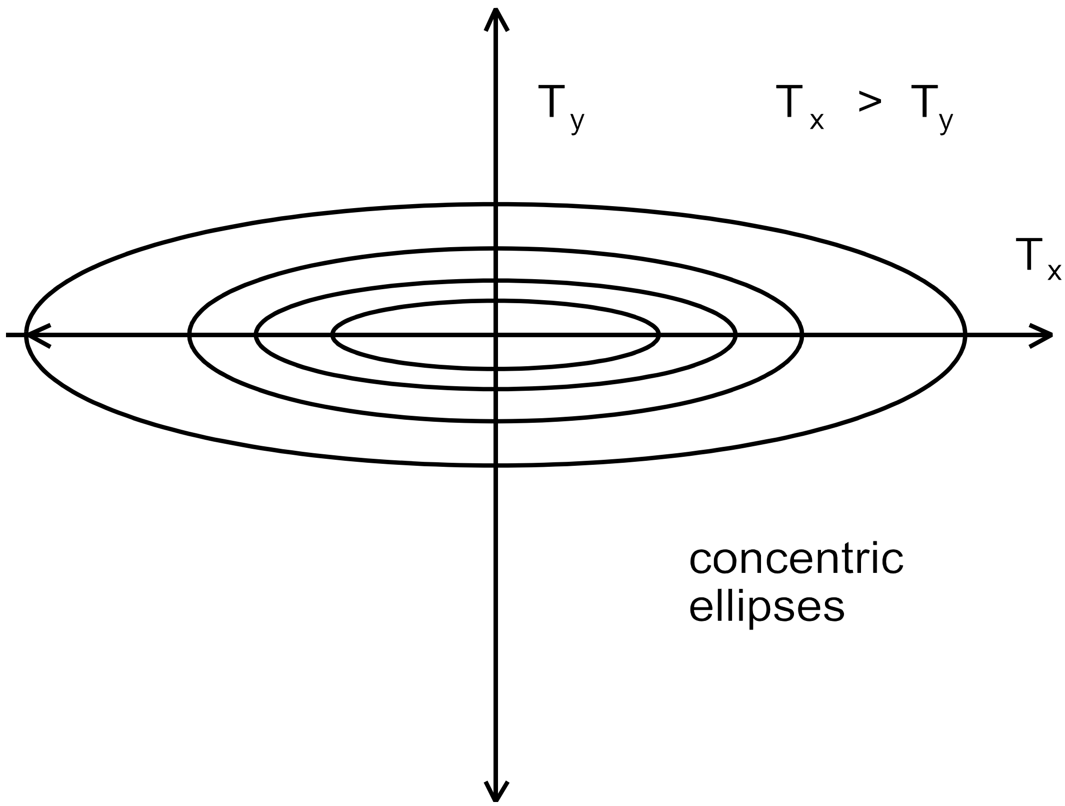
Solution to (4)

$$u_e = \frac{r_e^2 S_e}{4T_e t} = \frac{S}{4t}\left(\frac{T_x y^2 + T_y x^2 - 2T_{xy}xy}{T_x T_y - T_{xy}^2}\right)$$

$$s(r_e, t) = \frac{Q}{4\pi T_e} W(u_e)$$

The form of the solution is the same as in the homogeneous and isotropic aquifer case.

Drawdown Distribution



No longer an axial symmetric flow

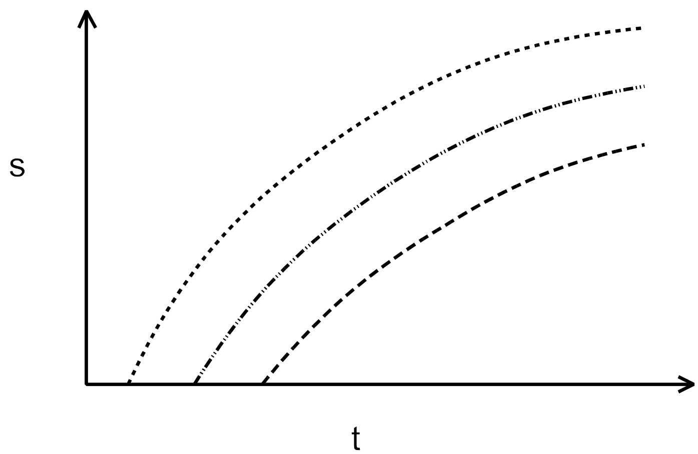
§ Parameter Estimation

Anisotropic Aquifers

- use at least three wells at different distances and directions

1) Method #1.

Drawdown-time data



$$T_e^2 = \left[\frac{Q}{4\pi s} W(u_e) \right]^2 \quad \rightarrow \text{This value should be the same for each well}$$

Use Theis' method to determine

$$\text{Well \#1} \;\; s_1 \;\; t_1 \rightarrow u_{e_1}, W(u_{e_1})$$
$$\text{Well \#2} \;\; s_2 \;\; t_2 \rightarrow u_{e_2}, W(u_{e_2})$$
$$\text{Well \#3} \;\; s_3 \;\; t_3 \rightarrow u_{e_3}, W(u_{e_3})$$

$$s(r_e, t) = \frac{Q}{4\pi T_e} W(u_e)$$

$$u_e = \frac{r_e^2 S_e}{4T_e t} = \frac{r_e^2}{4y} \frac{S}{T_e^2} = r_e^2 S = 4tT_e^2 u_e$$

$$S\left(T_x y^2 + T_y x^2 - 2T_{xy} xy\right) = 4u_e t\left(T_x T_y - T_{xy}^2\right)$$

Three wells:

$$W_1(x_1, y_1),\ W_2(x_2, y_2)\ \&\ W_3(x_3, y_3)$$

3 Equations and 3 unknowns

$$\begin{bmatrix} y_1^2 & x_1^2 & -2x_1 y_1 \\ y_2^2 & x_2^2 & -2x_2 y_2 \\ y_3^2 & x_3^2 & -2x_3 y_3 \end{bmatrix} \begin{bmatrix} ST_x \\ ST_y \\ ST_{xy} \end{bmatrix} = \begin{bmatrix} 4u_1 t_1 T_e^2 \\ 4u_2 t_2 T_e^2 \\ 4u_3 t_3 T_e^2 \end{bmatrix}$$

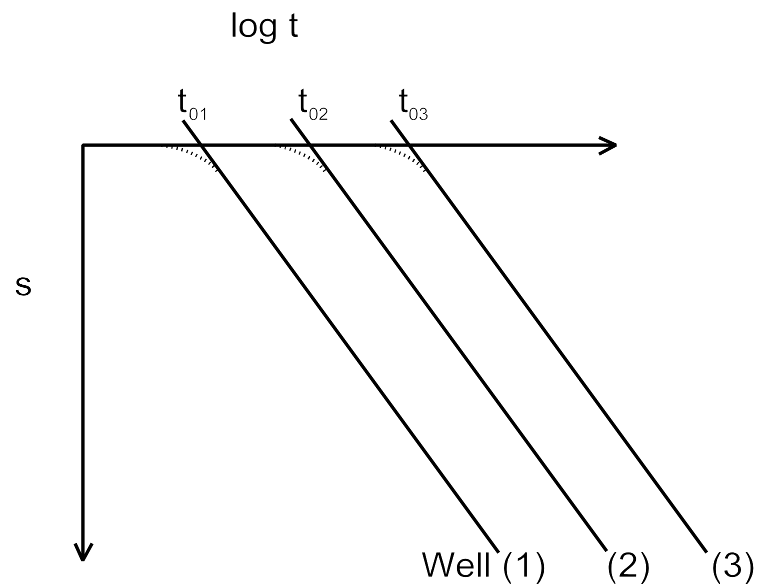
Solve for ST_x, ST_y, ST_{xy}

$$S^2 = \left(ST_x ST_y - S^2 T_{xy}^2\right) \Big/ \left(T_x T_y - T_{xy}^2\right)$$

Thus T_x, T_y, and T_{xy} are determined

Ref. Neuman et al., "Determination of horizontal aquifer anisotropy with three wells", *Ground Water* 22(1), 66-72, 1984

2. Jacob's straight line method. (Anisotropic Aquifer)



$$s(x,y,t) \cong \frac{2.303Q}{4\pi T_e}\log_{10}\left[\frac{2.25T_e^2}{\underbrace{\left(T_x y^2 + T_y x^2 - 2T_{xy}xy\right)}_{R^2}S}\right]$$

(1) Estimate T_e from s - t plot

$$T_e = \frac{0.183Q}{\Delta s} \qquad T_{e_1} \approx T_{e_2} \approx T_{e_3}$$

(2) Find t_0 for each well (where $s = 0$)
extrapolation

$$\begin{cases} \left(T_x y_1^2 + T_y x_1^2 - 2T_{xy} x_1 y_1\right) S = 2.25 T_e^2 t_{01} \\ \left(T_x y_2^2 + T_y x_2^2 - 2T_{xy} x_2 y_2\right) S = 2.25 T_e^2 t_{02} \\ \left(T_x y_3^2 + T_y x_3^2 - 2T_{xy} x_3 y_3\right) S = 2.25 T_e^2 t_{03} \end{cases}$$

$$\text{solve for } ST_x, ST_y, ST_{xy}$$

$$S^2 = \left(ST_x ST_y - S^2 T_{xy}^2\right) \Big/ T_e^2$$

Questions:

Again, if the aquifer is heterogenous, do you think the hydrographs from only three observation wells are adequate to delineate the average aquifer response that is defined in the solution? What would be the consistent approach to analyze the anisotropy?

Homework (Aquifer tests for nonleaky aquifers)

1. The saturated thickness of a nonleaky isotropic artesian aquifer infinite in areal extent is 100 feet. A production well fully penetrating the aquifer was continuously pumped at a constant rate of 1,500 g.p.m. for a period of one day. The drawdowns given in Table 1 were observed at a distance of 300 feet from the production well in a fully penetrating observation well. Compute the transmissivity, hydraulic conductivity, and storage coefficient of the aquifer.

Answer: T = 358,000 g.p.d./ft; P = 3,580 g.p.d./sq ft; S = 0.00047.

TABLE 1.

Time after pumping stared (min)	Drawdown (ft)
1	0.45
2	0.74
3	0.91
4	1.04
5	1.13
6	1.22
7	1.28
8	1.32
9	1.38
10	1.45
21	1.79
30	2.02
40	2.17
50	2.30
60	2.34
70	2.41
80	2.50
90	2.58
100	2.67
200	2.96
300	3.11
400	3.25
500	3.32
600	3.41
700	3.46
800	3.50
900	3.55
1,000	3.60

1,440	3.81

2. The transmissivity and storage coefficient of a nonleaky artesian aquifer infinite in areal extent are 1 x 10^5 g.p.d./ft and 3 x 10^{-4}, respectively. A fully penetrating production well has been discharging water at a constant rate of 1,000 g.p.m. for a period of 1 year. Compute the drawdowns at a distance of 10,000 feet from the production well for pumping periods of 10 days, 50 days, and 1 year and estimate the effective radius of the cone of depression for a pumping period of 1 year.
Answer: 2.7 ft; 4.4 ft; 6.8 ft; about 630,000 ft.