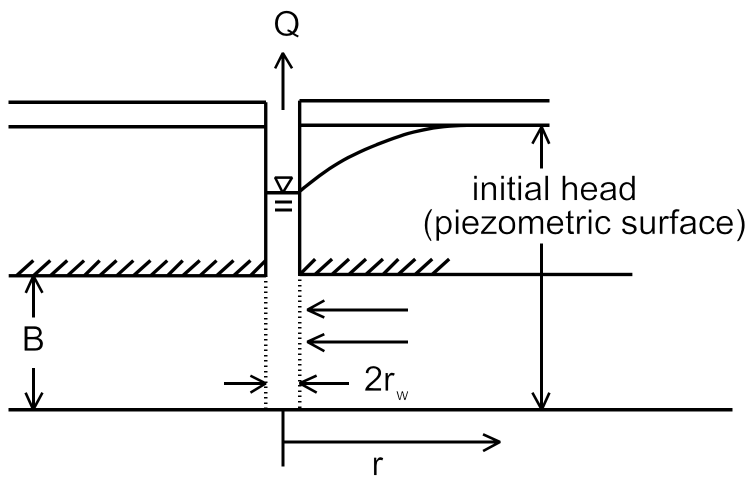
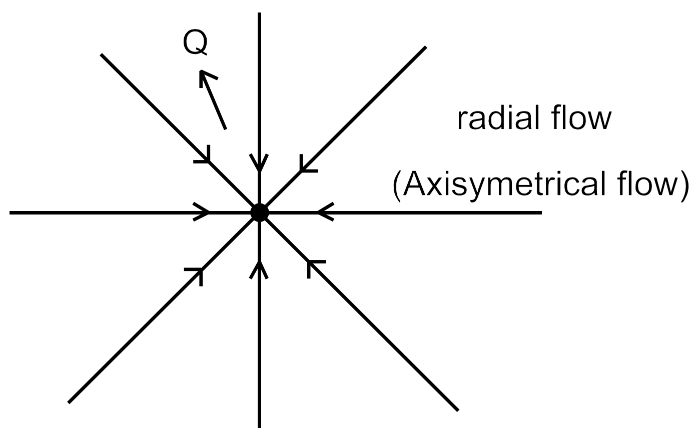
WELL HYDRAULICS

§ Steady Flow

1. flow to a well in an **ISOTROPIC** confined Aquifer





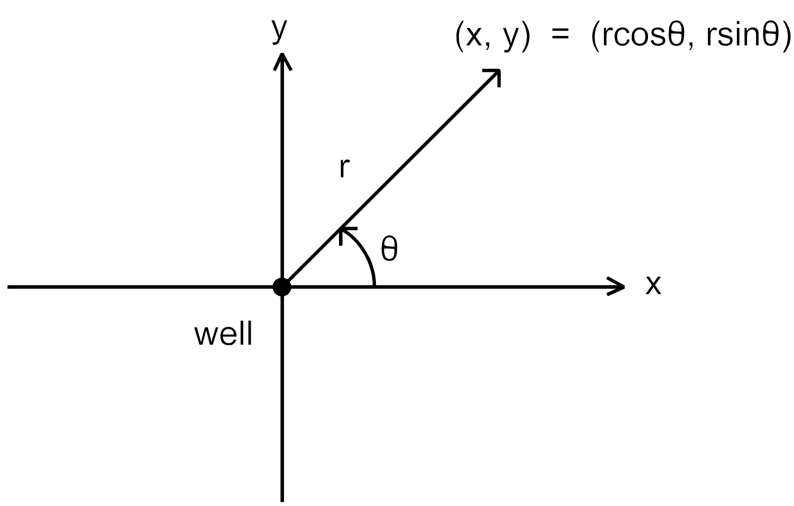
Assumptions

1. Isotropic, Homogeneous Aquifer
2. constant discharge Q
3. Horizontal flow
4. Infinite lateral extent

§ Governing P.D.E. for 2-D Horizontal Flow

$$T\frac{\partial^2 h}{\partial x^2} + T\frac{\partial^2 h}{\partial y^2} = 0 \qquad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

radial flow $\rightarrow$ polar coordinates



$$x = r\cos\theta$$

$$x = r\sin\theta \qquad r = \sqrt{\left(x^2 + y^2\right)}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\begin{aligned}
\frac{\partial^2 h}{\partial x^2} &= \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\cdot\frac{\partial r}{\partial x}\right)\cdot\frac{\partial r}{\partial x} \\
&= \frac{\partial}{\partial r}\left(\frac{\partial h}{\partial r}\cdot\frac{x}{r}\right)\cdot\frac{x}{r} \\
&= \frac{\partial^2 h}{\partial r^2}\left(\frac{x}{r}\right)^2 + \frac{x}{r}\frac{\partial h \partial (x/r)}{\partial r \partial x} \\
&= \frac{\partial^2 h}{\partial r^2}\left(\frac{x}{r}\right)^2 + \frac{\partial h}{\partial r}\left(\frac{1}{r} - \frac{x^2}{r^3}\right)
\end{aligned}$$

Similarly,

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 h}{\partial r^2}\left(\frac{y}{r}\right)^2 + \frac{\partial h}{\partial r}\left(\frac{1}{r} - \frac{y^2}{r^3}\right)$$

Now,

$$\begin{aligned}
\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} &= \left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right)\frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{\partial r}\left(\frac{2}{r} - \frac{x^2}{r^3} - \frac{y^2}{r^3}\right) \\
&= \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0
\end{aligned}$$

since $x^2 + y^2 = r^2$.

Therefore, the governing partial differential equation for steady radial flow in polar coordinates becomes

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0$$

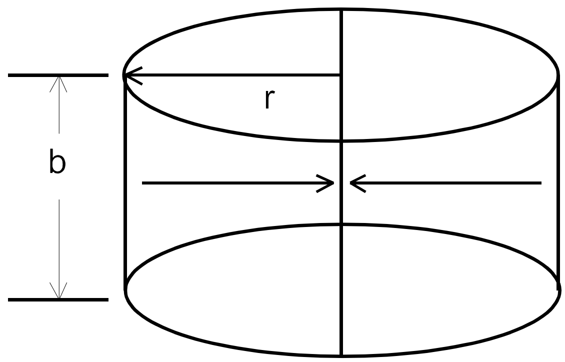
Notice that such a polar transformation reduces a two-dimensional problem to a one-dimensional problem, expressed in terms of r instead of x and y. As a result, solution to the equation becomes less cumbersome – even Jim Yeh can derive it.

Let:
$$\Phi = \frac{\partial h}{\partial r}$$
$$\frac{\partial \Phi}{\partial r} + \frac{1}{r}\Phi = 0$$

Solution
$$r\Phi = c_1$$
$$\frac{\partial h}{\partial r} = \frac{c_1}{r}$$

Thus, the general solution is
$$h(r) = c_1 \ln r + c_2$$

To evaluate the constant C_1, consider the inflow to a cylindrical surface around the pumping well at any distance r.
$$Q = KIA = K\frac{\partial h}{\partial r}2\pi rb = 2\pi rT\frac{\partial h}{\partial r}$$
$$\therefore\ r\frac{\partial h}{\partial r} = \frac{Q}{2\pi T} = c_1$$



$$h(r) = \frac{Q}{2\pi T} \ln r + c_2$$

If we say h at some $r = r_0$, is denoted as h_0

$$h(r_0) = h_0 = \frac{Q}{2\pi T} \ln r_0 + c_2$$

$$c_2 = h_0 - \frac{Q}{2\pi T} \ln r_0$$

$$\therefore\ h(r) = \frac{Q}{2\pi T} \ln r + h_0 - \frac{Q}{2\pi T} \ln r_0$$

Thus, the drawdown at any radial distance r is

$$s(r) = h_0 - h(r) = \frac{Q}{2\pi T} \ln(\frac{r_0}{r})$$

This is the so-called **Thiem's Equation** where r_0 is called the radius of influence. It is defined as the radial distance at which the drawdown due to pumping is insignificant.

Now, let's derive another solution for a different boundary condition. I.e.,

$$\text{say} \qquad h(r_w) = h_w \qquad \text{at} \qquad r = r_w$$

where r_w is the well radius. From the previous solution,

$$h(r) = \frac{Q}{2\pi T} \ln r + c_2$$

$$h_w = \frac{Q}{2\pi T} \ln r_w + c_2$$

$$c_2 = h_w - \frac{Q}{2\pi T} \ln r_w$$

$$\therefore\ h(r) = \frac{Q}{2\pi T} \ln r + h_w - \frac{Q}{2\pi T} \ln r_w$$

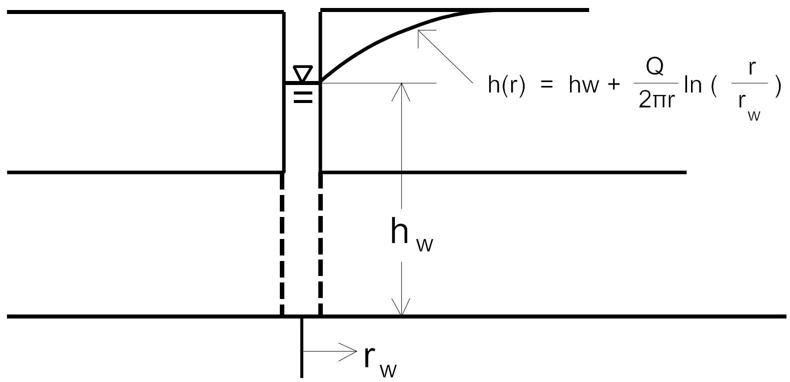
and

$$h(r) = h_w + \frac{Q}{2\pi T} \ln\left(\frac{r}{r_w}\right)$$

Remarks:

Based on Thiem's equation, we can conclude that

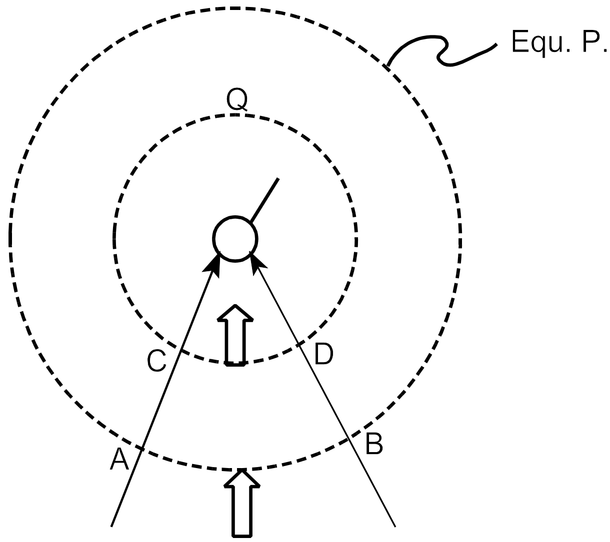
(1) The head distribution is logarithmic. The head decreases logarithmically toward the pumping well as illustrated in the figure below.



(2) In 3-D, the drawdown forms a cone of depression with a logarithmical profile.

Physical explanation:

1) converging flow
2) velocity high near the well
3) large gradient near the well



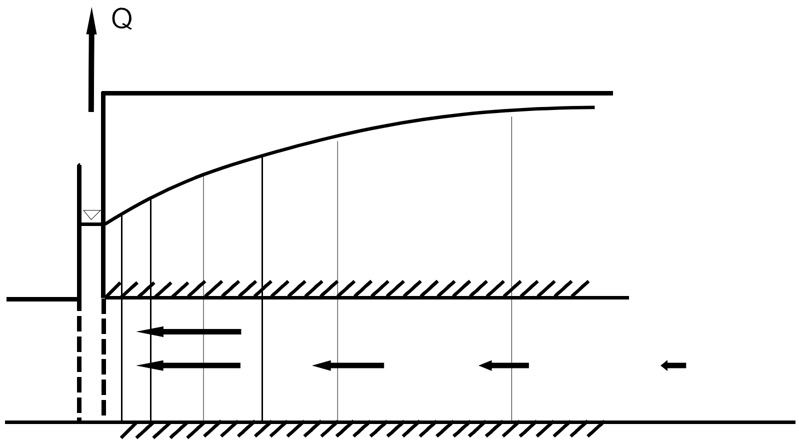
Steady flow $\Rightarrow$

$$Q_{\overline{CD}} = Q_{\overline{AB}}$$

$$W_{CD} T \frac{\partial h}{\partial r}\Big|_{CD} = W_{AB} T \frac{\partial h}{\partial r}\Big|_{AB}$$

$$T = T, \qquad W_{CD} < W_{AB}$$

$$\frac{\partial h}{\partial r}\Big|_{CD} > \frac{\partial h}{\partial r}\Big|_{AB}$$



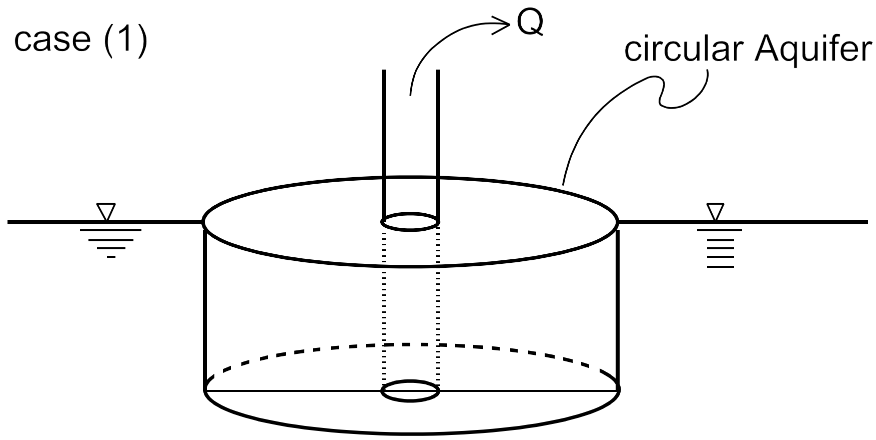
(3)

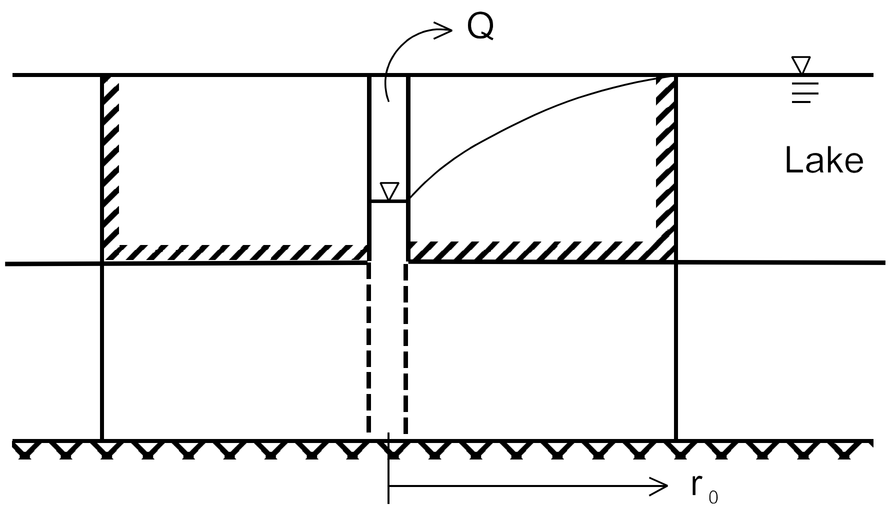
$$h(r) = h_w + \frac{Q}{2\pi T}\ln\left(\frac{r}{r_w}\right)$$

$$Let \qquad r = \infty \qquad\qquad h(\infty) = h_w + \infty = \infty$$

Based on the Thiem equation, the head at $r = \infty$ must be infinitely large. This is physically impossible unless you live in the deep space nine. So, why does the Thiem equation produce such an erroneous solution? Before answering this question, we should ask ourselves: Is the assumption of steady flow to a well in a confined aquifer of infinite lateral extent reasonable? Let's examine several different cases of flow to a well.

Case (1) Flow to a well in a circular aquifer surrounded by a lake.


case (1)
Q
circular Aquifer

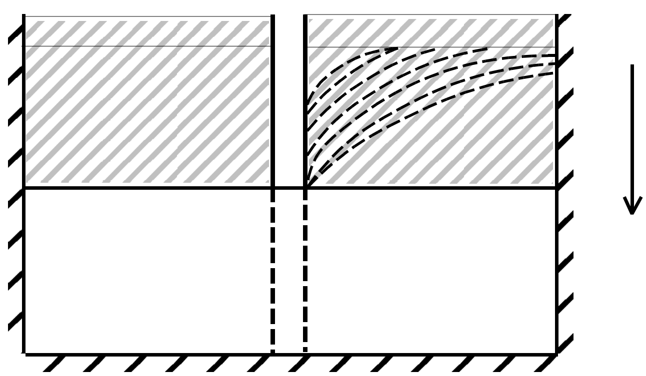

Q
Lake
r 0

The head at the lake, h_0, must be constant if the lake is large enough to supply water for the pumpage. As the pumping starts, the cone of depression grows outward to the lake. During this period of time, all the water pumped must come from the storage of the aquifer (compaction of the aquifer and expansion of water). Once the cone of depression reaches the lake, water discharged from the well is syphoned directly from the lake. The aquifer no longer has to supply the water from the storage and thus the pressure head in the aquifer must be steady. In addition, the head in the aquifer near the lake must be equal to the water level in the lake. Therefore, steady flow exists in this case. Then, the head distribution along *r*, based on Thiem's equation, is

$$h(r) = h_0 + \frac{Q}{2\pi T} \ln\left(\frac{r}{r_0}\right)$$

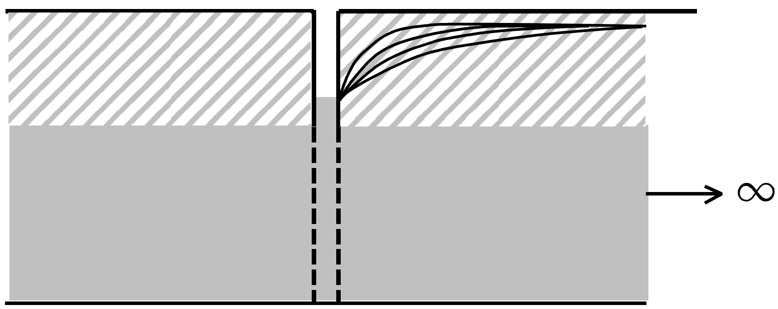
At $r = r_0$, $h(r_0) = h_0$. The infinite head problem as mentioned earlier no longer exists.

Case (2)Flow to a well in a circular confined aquifer surrounded by an impermeable boundary



Since this aquifer is surrounded by an impermeable boundary, when the drawdown reaches the impermeable boundary, no external source is available to supply the pumpage. As a result, the aquifer is not replenished and water must continuously come from the aquifer storage due to expansion of water and compression of the porous media. Drawdown will continue until the groundwater reservoir is completely depleted. We do not expect steady state flow in this aquifer.

Case (3) Flow to a well in an aquifer of infinite lateral extent



In this case, the cone of depression grows continuously in lateral directions to release enough water from aquifer storage to supply the pumpage. As the cone of depression grows, the volume of the aquifer that releases water increases. Therefore, the amount of pressure drop required to supply the pumpage at the far field becomes smaller. In practice, the drawdown at the far field may be small and negligible but theoretically the drawdown exists. Consequently, the cone of depression continuously grows outward and steady-state flow does not exist. If this is true in the aquifer with infinite lateral extent, why should one use the Thiem equation that is good for steady flow only to predict the head distribution? This is the reason why that the head predicted by the equation is physically impossible at large distances.

Conclusions:

Thiem's equation is not valid for confined aquifers with infinite lateral extent.

1) It is valid only in the close proximity of a well where steady flow seems present.

2) It is valid if the aquifer has a recharge boundary.

2. Flow to a well in an Anisotropic Aquifer

The governing partial differential equation for flow in an anisotropic aquifer is

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = 0$$

if we assume that our *x-y* coordinate system aligns with the principal directions of the transmissivity. Before digging into the mathematical solution to this equation, let's guess what the solution will be. Based on your common sense God gives you, you probably say that the cone of depression will not be a concentric, circular black hole any more but an elliptical one. Yeah, smart ass but tell me why.

Black or white? It depends on the microchip God implanted in your eyes. Well, can we visualize a elliptical cone of depression as a circular one? Sure, it's simple. If you have big blue round eyes, squeeze them so that it has the same aspect ratio as the ellipse and you will see it as a perfect circle. If that is the case, then we have already had the solution for the equation of flow in an anisotropic aquifer, haven't we? God, I hope you are not that stupid and take what I said seriously. What I mean by squeezing your eye balls is simply to rescale your coordinates. That is,

This means that distances in the direction of the greater T are scaled.

if $T_x \neq T_y$, we choose $x' = x\sqrt{T_y/T_x}$

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial x'}\frac{\partial x'}{\partial x} = \sqrt{T_y/T_x}\,\frac{\partial h}{\partial x'}$$

$$\frac{\partial^2 h}{\partial x^2} = \sqrt{T_y/T_x}\,\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x'}\right)$$

$$= \sqrt{T_y/T_x}\,\frac{\partial}{\partial x'}\left(\frac{\partial h}{\partial x'}\right)\frac{\partial x'}{\partial x} = \frac{T_y}{T_x}\frac{\partial^2 h}{\partial x'^2}$$

therefore $$T_x \frac{\partial^2 h}{\partial x^2} = T_y \frac{\partial^2 h}{\partial x'^2}$$

$$R^2 = x'^2 + y^2$$

$$= x^2 \frac{T_y}{T_x} + y^2$$

The original equation in terms of T_x, T_y, x, and y becomes

$$\frac{\partial^2 h}{\partial x'^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

This equation now is the same as the governing equation for flow in isotropic aquifers which can be easily rewritten in a polar coordinate system. We have already derived the solution for this equation. However, notice that the x-coordinate has been scaled. Can you derive solutions for steady-state flow to a well in an anisotropic aquifer by yourself now?